

THE RECURSIVE NATURE OF COMINUSCULE SCHUBERT CALCULUS

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ABSTRACT. The necessary and sufficient Horn inequalities which determine the non-vanishing Littlewood-Richardson coefficients in the cohomology of a Grassmannian are recursive in that they are naturally indexed by non-vanishing Littlewood-Richardson coefficients on *smaller* Grassmannians. We show how non-vanishing in the Schubert calculus for cominuscule flag varieties is similarly recursive. For these varieties, the non-vanishing of products of Schubert classes is controlled by the non-vanishing products on smaller cominuscule flag varieties. In particular, we show that the lists of Schubert classes whose product is non-zero naturally correspond to the integer points in the feasibility polytope, which is defined by inequalities coming from non-vanishing products of Schubert classes on smaller cominuscule flag varieties. While the Grassmannian is cominuscule, our necessary and sufficient inequalities are different than the classical Horn inequalities.

INTRODUCTION

We investigate the following general problem: Given Schubert subvarieties X, X', \dots, X'' of a flag variety, when is the intersection of their general translates

$$(1) \quad gX \cap g'X' \cap \dots \cap g''X''$$

non-empty? When the flag variety is a Grassmannian, it is known that such an intersection is non-empty if and only if the indices of the Schubert varieties, expressed as partitions, satisfy the linear Horn inequalities. The Horn inequalities are themselves indexed by lists of partitions corresponding to such non-empty intersections on smaller Grassmannians. This recursive answer to our original question is a consequence of work of Klyachko [15] who linked eigenvalues of sums of hermitian matrices, highest weight modules of \mathfrak{sl}_n , and the Schubert calculus, and of Knutson and Tao's proof [16] of Zelevinsky's Saturation Conjecture [28]. These two results proved Horn's Conjecture [12] about the eigenvalues of sums of Hermitian matrices. This had wide implications in mathematics (see the surveys [7, 8]) and raised many new and evocative questions. For example, the recursive nature of this geometric question concerning the intersection of Schubert varieties was initially mysterious, as the proofs used much more than the geometry of the Grassmannian.

Belkale [2] provided a geometric proof of the Horn inequalities, which explains their recursive nature. His method relied upon an analysis of the tangent spaces to Schubert varieties. One of us (Purbhoo) reinterpreted Belkale's proof [20] using two-step partial flag

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varieties (Grassmannians are one-step partial flag varieties) for the general linear group. This approach starts from the observation that the non-emptiness of an intersection (1) can be translated into a question of transversality involving the tangent spaces of Schubert varieties (Proposition 9).

For other groups, two-step partial flag varieties are replaced by fibrations of flag varieties. Suppose that $R \subset P$ are parabolic subgroups of a complex reductive algebraic group G . Then $P/R = L/Q$, where L is the Levi subgroup of P and Q is a parabolic subgroup of P and we have the fibration sequence of flag varieties.

$$\begin{array}{ccc} L/Q = P/R & \longrightarrow & G/R \\ & & \downarrow \\ & & G/P \end{array}$$

Given Schubert varieties X on G/P and Y on L/Q , there is a unique lifted Schubert variety Z on G/R which maps to X with fiber Y over the generic point of X . Each tangent space of G/R has a map to \mathfrak{z} , the dual of the center of the nilradical of R . Let $C(X, Y)$ be the codimension in \mathfrak{z} of the image of the tangent space to Z at a smooth point.

Suppose that we have Schubert varieties X, X', \dots, X'' of G/P such that the intersection (1) of their general translates is non-empty. Given Schubert varieties Y, Y', \dots, Y'' of L/Q whose general translates (by elements of L) have non-empty intersections, then we have the inequality

$$(2) \quad C(X, Y) + C(X', Y') + \dots + C(X'', Y'') \leq \dim \mathfrak{z}.$$

We show that a subset of these necessary inequalities are sufficient to determine when a general intersection (1) of Schubert varieties is non-empty, when G/P is a **cominuscule flag variety**. For each cominuscule G/P , we identify a set $M(P)$ of parabolic subgroups $Q \subset L$. We state a version of our main result (Theorem 4).

Theorem. *Suppose that G/P is a cominuscule flag variety. Then the intersection (1) is non-empty if and only if for every $Q \in M(P)$ and every Schubert varieties Y, Y', \dots, Y'' of L/Q whose general translates have non-empty intersection, the inequality (2) holds.*

As discussed in Section 2, this solves the question of when an arbitrary product of Schubert classes on a cominuscule flag variety is non-zero.

The subgroups $Q \in M(P)$ have the property that L/Q is also cominuscule, and thus the inequalities which determine the non-emptiness of (1) are recursive in that they come from similar non-empty intersections on smaller cominuscule flag varieties. For Grassmannians, these inequalities are different than the Horn inequalities, and hence give a new proof of the Saturation Conjecture. Moreover, the inequalities for the Lagrangian and orthogonal Grassmannians are different, despite their having the same sets of solutions!

By cominuscule flag variety, we mean the orbit of a highest weight vector in (the projective space of) a cominuscule representation of a linear algebraic group G . These are analogs of the Grassmannian for other Lie types; their Bruhat orders are distributive lattices [19] and the multiplication in their cohomology rings is governed by a uniform Littlewood-Richardson rule [27]. Cominuscule flag varieties G/P are distinguished in that the unipotent radical of P is abelian [22] and in that a Levi subgroup L of P acts on

the tangent space at eP with finitely many orbits. There are other characterizations of cominuscule flag varieties which we discuss in Section 1.4. We use that G/P is cominuscule in many essential ways in our arguments, which suggests that cominuscule flag varieties are the natural largest class of flag varieties for which these tangent space methods can be used to study the non-vanishing of intersections (1).

Since the algebraic groups G and L need not have the same Lie type, in many cases the necessary and sufficient inequalities of Theorem 4 are indexed by non-empty intersections of Schubert varieties on cominuscule flag varieties of a different type. For example, the inequalities for the Lagrangian Grassmannian are indexed by non-empty intersections on ordinary Grassmannians. This is in contrast to the classical Horn recursion, which is purely in type A , involving only ordinary Grassmannians. Thus the recursion we obtain is a recursion within the class of cominuscule flag varieties, rather than a type-by-type recursion. This is reflected in our proof of the cominuscule recursion, which is entirely independent of type; in particular we do not appeal to the classification of cominuscule flag varieties.

This paper is structured as follows. Section 1 establishes our notation and develops background material. Section 2 states our main theorem precisely (Theorem 4) and derives necessary inequalities (Theorem 2), which are more general than the inequalities (2). Section 3 contains the proof of our main theorem, some of which relies upon technical results about root systems, which are given in the Appendix. In Section 4 we examine the cominuscule recursion in more detail, describing it on a case-by-case basis. In Section 5 we compare our results and inequalities to other systems of inequalities for non-vanishing in the Schubert calculus, including the classical Horn inequalities, and the dimension inequalities of Belkale and Kumar [3]. We have attempted to keep Sections 3 and 4 independent, so that the reader who is more interested in examples may read them in the opposite order.

1. DEFINITIONS AND OTHER BACKGROUND MATERIAL

We review basic definitions and elementary facts that we use concerning linear algebraic groups, Schubert varieties and their tangent spaces, transversality, and cominuscule flag varieties. All algebraic varieties, groups, and algebras will be over the complex numbers, as the proofs we give of the main results are valid only for complex varieties.

1.1. Linear algebraic groups and their flag varieties. We assume familiarity with the basic theory of algebraic groups and Lie algebras as found in [5, 10, 13, 24]. We use capital letters $B, G, H, K, L, P, Q, R, \dots$ for algebraic groups and the corresponding lower-case fraktur letters for their Lie algebras $\mathfrak{b}, \mathfrak{g}, \mathfrak{h}, \mathfrak{k}, \mathfrak{l}, \mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \dots$. We also use lower-case fraktur letters $\mathfrak{s}, \mathfrak{z}$ for subquotients of these Lie algebras. Throughout, G will be a reductive algebraic group, P a parabolic subgroup of G , $B \subset P$ a Borel subgroup of G , and $e \in G$ will be the identity. Let H be a maximal torus of G with $H \subset B$. Let $L \subset P$ be the Levi (maximal reductive) subgroup containing H . We have the Levi decomposition $P = LN_P$ of P with N_P its (H -stable) unipotent radical. Write G^{ss} and L^{ss} for the semisimple parts of G and L , respectively. Write W or W_G for the Weyl group of G , which is the quotient $N_G(H)/H$. Note that $W_P = W_L$.

There is a dictionary between parabolic subgroups Q of L and parabolic subgroups R of P which contain a maximal torus of L ,

$$Q = R \cap L \quad \text{and} \quad R = QN_P.$$

Thus R is the maximal subgroup of P whose restriction to L is Q . We will always use the symbols Q and R for parabolic subgroups of L and P associated in this way. We will typically have $H \subset Q(\subset R)$. Set $B_L := B \cap L$, a Borel subgroup of L that contains H . We say that Q and R are *standard parabolic subgroups* if $B_L \subset Q$ (equivalently $B \subset R$). Then the surjection $\text{pr}: G/R \rightarrow G/P$ has fiber $P/R = L/Q$, so we have the fibration diagram:

$$(3) \quad \begin{array}{ccc} L/Q = P/R & \longrightarrow & G/R \\ & & \downarrow \text{pr} \\ & & G/P \end{array}$$

Let $\Phi \subset \mathfrak{h}^*$ be the roots of the Lie algebra \mathfrak{g} . These decompose into positive and negative roots, $\Phi = \Phi^+ \sqcup \Phi^-$, where Φ^- are the roots of \mathfrak{b} . Our convention that the roots of B are negative will simplify the statements of our results. Write Δ for the basis of simple roots in Φ^+ . For $\alpha \in \Phi$, let \mathfrak{g}_α be the (1-dimensional) α -weight space of \mathfrak{g} . Then we have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha.$$

We write $\Phi(\mathfrak{s})$ for the non-zero weights of an H -invariant subquotient \mathfrak{s} of \mathfrak{g} , and $\Phi^+(\mathfrak{s})$ for $\Phi(\mathfrak{s}) \cap \Phi^+$. Note that these weights are all roots of \mathfrak{g} . The Killing form on \mathfrak{g} pairs \mathfrak{g}_α with $\mathfrak{g}_{-\alpha}$ and identifies \mathfrak{g} with its dual. Under this identification, the dual \mathfrak{s}^* of an H -invariant subquotient \mathfrak{s} is another subquotient of \mathfrak{g} , and $\Phi(\mathfrak{s}^*) = -\Phi(\mathfrak{s})$. In this way, the dual of \mathfrak{n}_P is identified with $\mathfrak{g}/\mathfrak{p}$.

The Weyl group W acts on all these structures. For example, if $g \in N(H)$, then the conjugate gBg^{-1} of B depends only upon the coset gH , which is the element w of W determined by g . Write wBw^{-1} for this conjugate, and use similar notation for conjugates of other subgroups of G . Conjugation induces a left action on roots and we have $w\Phi^- = \Phi(wBw^{-1})$. The *inversion set* of $w \in W$ is the set of positive roots which become negative under the action of w , $\text{Inv}(w) := w^{-1}\Phi^- \cap \Phi^+$. The inversion set determines w , and the cardinality of $\text{Inv}(w)$ is the *length* of w , $\ell(w) := |\text{Inv}(w)|$.

Borel subgroups containing H are conjugate by elements of W . For $w \in W_G$, $wBw^{-1} \cap P$ is a solvable subgroup of P which is not necessarily maximal. However, $wBw^{-1} \cap L$ is a Borel subgroup of L , and this has a nice description in terms of the Weyl groups W_G and $W_L = W_P$. Let $\pi \in wW_L$ be the coset representative of minimal length (with respect to reflections in the simple roots Δ). Write W^P for this set of minimal length coset representatives, and similarly write W^Q for the set of minimal length representatives of cosets of W_Q in W_L . Set $\lambda := \pi^{-1}w \in W_L$. Then $\ell(w) = \ell(\lambda) + \ell(\pi)$. This corresponds to a decomposition of the inversion set of w . Note that $\Phi^+ = \Phi^+(\mathfrak{l}) \sqcup \Phi(\mathfrak{g}/\mathfrak{p})$. Then

$$(4) \quad \begin{aligned} \text{Inv}(\lambda) &= \text{Inv}(w) \cap \Phi(\mathfrak{l}), \\ \text{Inv}(\pi) &= \lambda \text{Inv}(w) \cap \Phi(\mathfrak{g}/\mathfrak{p}), \quad \text{and} \\ \text{Inv}(w) &= \text{Inv}(\lambda) \sqcup \lambda^{-1} \text{Inv}(\pi). \end{aligned}$$

1.2. Schubert varieties and their tangent spaces. Points of the flag variety G/P are parabolic subgroups conjugate to P , with gP corresponding to the subgroup gPg^{-1} . A Borel subgroup B of G acts with finitely many orbits on G/P . When $H \subset B \subset P$, each orbit has the form BwP for some $w \in W$. The coset wP is the unique H -fixed point in the orbit BwP .

If $wW_P = w'W_P$ for some $w, w' \in W$, then $wP = w'P$. Thus these B -orbits are indexed by the set W^P . If $P' \in B\pi P$ for $\pi \in W^P$, then we say that P' has *Schubert position* π with respect to the Borel subgroup B . When this happens, there is a $b \in B$ such that $bP'b^{-1} \supset \pi B\pi^{-1}$. The decomposition

$$G/P = \coprod_{\pi \in W^P} B\pi P$$

of G/P into B -orbits is the Bruhat decomposition of G/P . The orbit $X_\pi^\circ B := B\pi P$ is called a *Schubert cell* and is parametrized by the unipotent subgroup $B \cap \pi N_{P^\circ} \pi^{-1}$, where N_{P° is the unipotent radical of the parabolic subgroup P° opposite to P . The closure of $X_\pi^\circ B$ is the *Schubert variety* $X_\pi B$, which has dimension $\ell(\pi)$.

For each $\pi \in W^P$, define the *planted Schubert cell* X_π° to be the translated orbit $\pi^{-1}B\pi P$, and the *planted Schubert variety* X_π to be its closure. A translate of the Schubert cell $X_\pi^\circ B$ contains eP if and only if it has the form pX_π° for some $p \in P$.

The tangent space to G/P at eP is naturally identified with the Lie algebra quotient $\mathfrak{g}/\mathfrak{p}$. As the nilpotent subgroup N_{P° parameterizes G/P in a neighborhood of eP , the tangent space can also be identified with \mathfrak{n}_{P° ; indeed, the natural map, $\mathfrak{n}_{P^\circ} \rightarrow \mathfrak{g}/\mathfrak{p}$, is an H -equivariant isomorphism. Since X_π° is parametrized by $\pi^{-1}B\pi \cap N_{P^\circ}$, its tangent space T_π at eP (an H -submodule of \mathfrak{n}_{P°) has weights

$$\begin{aligned} \Phi(T_\pi) &= (\pi^{-1}\Phi^-) \cap \Phi(\mathfrak{n}_{P^\circ}) \\ &= (\pi^{-1}\Phi^-) \cap \Phi(\mathfrak{g}/\mathfrak{p}) = \text{Inv}(\pi). \end{aligned}$$

P acts on the tangent space $T_{eP}G/P = \mathfrak{g}/\mathfrak{p}$. Translating $T_\pi \subset \mathfrak{g}/\mathfrak{p}$ by $p \in P$, we obtain the tangent space pT_π to pX_π at eP .

These planted Schubert varieties and their tangent spaces fit into the fibration diagram (3). Let $R \subset P$ be standard parabolic subgroups of G , and set $Q := L \cap R$ be the standard parabolic subgroup of L corresponding to R . Minimal coset representatives of W_R in W_G are products $\pi\lambda$, where $\pi \in W^P$ and $\lambda \in W^Q$ is a minimal representative of W_Q in W_L . Then the image of the Schubert cell $B\pi\lambda R$ of G/R in G/P is the Schubert cell $B\pi P$. When π is the identity, we have that $B\lambda R/R = B_L\lambda Q/Q$.

In general, the fiber $B\pi\lambda R \rightarrow B\pi P$ is isomorphic to $B_L\lambda Q$. In particular, we have

$$\begin{array}{ccc} X_\lambda^\circ & \longrightarrow & X_{\pi\lambda}^\circ \\ & & \downarrow \text{pr} \\ & & \lambda^{-1}X_\pi^\circ \end{array}$$

and thus we obtain a short exact sequence of the tangent spaces

$$(5) \quad T_\lambda \hookrightarrow T_{\pi\lambda} \twoheadrightarrow \lambda^{-1}T_\pi.$$

Indeed, if $bRb^{-1} \in X_{\pi\lambda}^\circ$ lies in the fiber, then b lies in $\lambda^{-1}\pi^{-1}B\pi\lambda \cap P$. Since R contains the unipotent radical of P , we can assume that in fact

$$b \in \lambda^{-1}\pi^{-1}B\pi\lambda \cap L = \lambda^{-1}B_L\lambda$$

and thus $bRb^{-1} \cap L = bQb^{-1} \in X_\lambda^\circ$. The converse is straightforward. Here, we used that $\pi^{-1}B\pi \cap L = B_L$, which follows from $\text{Inv}(\pi) \cap \Phi(\mathfrak{l}) = \emptyset$.

1.3. Transversality. We write V^* for the linear dual of a vector space V and write U^{ann} for the annihilator of a subspace U of V . A collection of linear subspaces of V *meets transversally* if their annihilators are in direct sum.

For us, a variety will always mean a reduced, but not-necessarily irreducible scheme over the complex numbers. A collection of algebraic subvarieties of a smooth variety X is *transverse at a point p* if they are each smooth at p and if their tangent spaces at p meet transversally, as subspaces of the tangent space of X at p . A collection of algebraic subvarieties of a smooth variety X *meets transversally* if they are transverse at the generic point of every component in their intersection. We freely invoke Kleiman's Transversality Theorem [14], which asserts that if a (complex) reductive algebraic group acts transitively on a smooth variety X , then general translates of subvarieties of X meet transversally.

We establish the following result from elementary linear algebra, which will be indispensable in analyzing the transversality of Schubert varieties.

Proposition 1. *Suppose that we have a short exact sequence of vector spaces*

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0.$$

Let U_1, \dots, U_s be linear subspaces of V and set $S_i := W \cap U_i$ and $M_i := (S_i + W)/W$, for $i = 1, \dots, s$.

- (i) *If U_1, \dots, U_s are transverse in V , then M_1, \dots, M_s are transverse in V/W .*
- (ii) *If S_1, \dots, S_s are transverse in W , then U_1, \dots, U_s are transverse if and only if M_1, \dots, M_s are transverse.*

Proof. It suffices to prove this for $s = 2$, as subspaces are transverse if and only if they are pairwise transverse. If $U_1^{\text{ann}}, U_2^{\text{ann}}$ form a direct sum, then their subspaces $M_1^{\text{ann}}, M_2^{\text{ann}}$ form a direct sum, and (i) follows immediately. This proves one direction of (ii). For the other, consider its dual statement: If $S_1^{\text{ann}} + S_2^{\text{ann}}$ and $M_1^{\text{ann}} + M_2^{\text{ann}}$ are direct sums, then so is $U_1^{\text{ann}} + U_2^{\text{ann}}$. Note that $M_i^{\text{ann}} = U_i^{\text{ann}} \cap (V/W)^*$ and S_i^{ann} is the image of U_i^{ann} in W^* . But if $U_1^{\text{ann}} + U_2^{\text{ann}}$ is not a direct sum, then $U_1^{\text{ann}} \cap U_2^{\text{ann}} \neq \{0\}$. By assumption on M_1^{ann} and M_2^{ann} , the image of $U_1^{\text{ann}} \cap U_2^{\text{ann}}$ in W^* is a non-empty subspace lying in $S_1^{\text{ann}} \cap S_2^{\text{ann}}$. \square

It follows immediately from the definition of transversality that if U_1, \dots, U_s are transverse linear subspaces of V , then we must have the codimension inequality

$$\sum_{i=1}^s \text{codim } U_i \leq \dim V.$$

We freely make use of this basic fact in our arguments.

1.4. Cominuscule flag varieties. We list several equivalent characterizations of cominuscule flag varieties G/P . Recall that $P = LN_P$ is the Levi decomposition of P . Then

- (i) N_P is abelian.
- (ii) L has finitely many orbits on N_P , equivalently on its Lie algebra \mathfrak{n}_P and on $\mathfrak{g}/\mathfrak{p} = T_{eP}G/P$.
- (iii) $\mathfrak{g}/\mathfrak{p}$ is an irreducible representation of L , which implies that the Weyl group W_L acts transitively on roots of the same length in $\Phi(\mathfrak{g}/\mathfrak{p})$.
- (iv) $P = P_\alpha$ is a maximal parabolic subgroup of G and the omitted simple root α occurs with coefficient 1 in the highest root of G .

Sources for these equivalences, with references, may be found in [18, 22, 23]. Cominuscule flag varieties come in five infinite families with two exceptional cominuscule flag varieties.

Let G/P be a cominuscule flag variety and α the root corresponding to the maximal parabolic subgroup P . As explained in [18], the semisimple part L^{ss} of the Levi subgroup of P has Dynkin diagram obtained from that of G by deleting the node corresponding to the root α . The representation of L^{ss} on the tangent space $\mathfrak{g}/\mathfrak{p}$ is the tensor product of fundamental representations given by marking the nodes in the diagram of L^{ss} that were adjacent to α . This is summarized in Table 1.

G/P	$\mathrm{Gr}(k, n+1)$	Q^{2n-1}	$LG(n)$	Q^{2n-2}
G	A_n	B_n	C_n	D_n
α				
L^{ss}	$A_{k-1} \times A_{n-k}$	B_{n-1}	A_{n-1}	D_{n-1}
$\mathfrak{g}/\mathfrak{p}$				

G/P	$OG(n)$	\mathbb{OP}^2	$G_\omega(\mathbb{O}^3, \mathbb{O}^6)$
G	D_n	E_6	E_7
α			
L^{ss}	A_{n-1}	D_5	E_6
$\mathfrak{g}/\mathfrak{p}$			

TABLE 1. Cominuscule Flag Varieties

The varieties Q^{2n-1} and Q^{2n-2} are odd- and even-dimensional quadrics respectively. $LG(n)$ is the Lagrangian Grassmannian. The superscript 2 in the Dynkin diagram of A_{n-1} in the column for $LG(n)$ indicates that this representation has highest weight twice the corresponding fundamental weight. The second cominuscule flag variety in type D_n is the *orthogonal Grassmannian*, $OG(n)$. This is one of two components of the space of maximal isotropic subspaces in the vector space \mathbb{C}^{2n} , which is equipped with a nondegenerate symmetric bilinear form. It is also known as the spinor variety. The notation \mathbb{OP}^2 is for the

Cayley plane (projective plane for the octonians) and $G_\omega(\mathbb{O}^3, \mathbb{O}^6)$ is borrowed from [18] (as was the idea for Table 1).

2. FEASIBILITY AND STATEMENT OF MAIN THEOREM

The general problem that we are investigating is, given π_1, \dots, π_s with $\pi_i \in W^P$ and general translates $g_1 X_{\pi_1} B, \dots, g_s X_{\pi_s} B$ of the corresponding Schubert varieties, when is the intersection

$$(6) \quad g_1 X_{\pi_1} B \cap g_2 X_{\pi_2} B \cap \dots \cap g_s X_{\pi_s} B$$

non-empty? A list π_1, \dots, π_s with $\pi_i \in W^P$ is a *Schubert position* for G/P . It is *feasible* if such general intersections (6) are non-empty. For $g \in G$, the translate $gX_\pi B$ is another Schubert variety, but for the Borel subgroup gBg^{-1} . Thus, π_1, \dots, π_s is a feasible Schubert position if, for any Borel subgroups B_1, \dots, B_s , there is a parabolic subgroup P' having Schubert position π_i with respect to B_i for each $i = 1, \dots, s$.

Feasibility is often expressed in terms of algebra. Write σ_π for the class of a Schubert variety $X_\pi B$ in the cohomology ring of G/P . Then the product $\prod_{i=1}^s \sigma_{\pi_i}$ is non-zero if and only if a general intersection of the form (6) is non-empty, if and only if the Schubert position π_1, \dots, π_s is feasible. If $\sum_{i=1}^s \text{codim } X_{\pi_i} B = \dim G/P$, then the generic intersection (6) is finite, and the integral

$$\int_{G/P} \sigma_{\pi_1} \sigma_{\pi_2} \cdots \sigma_{\pi_s}$$

computes the number of points in this intersection. In this case we say that π_1, \dots, π_s is a *top-degree* Schubert position.

In this section, we state two theorems, Theorem 2 and our main result, Theorem 4, which give conditions for feasibility in terms of inequalities. We then show how the problem of feasibility can be reformulated in terms of transversality for tangent spaces to Schubert varieties. Using this, we prove Theorem 2. The ideas in this section form the foundation for the proof of Theorem 4, which is given in Section 3.

2.1. Statement of main results. As in Section 1.2, let $R \subset P$ be standard parabolic subgroups of G , and let $Q := L \cap R$. Let \mathfrak{s} be any R -submodule of the nilradical \mathfrak{n}_R of \mathfrak{r} . As \mathfrak{n}_R^* is identified with the tangent space to G/R at eR , dual to the inclusion $\mathfrak{s} \hookrightarrow \mathfrak{n}_R$ is the surjection

$$\varphi_{\mathfrak{s}} : T_{eR}G/R \twoheadrightarrow \mathfrak{s}^*.$$

Theorem 2. *Suppose that π_1, \dots, π_s is a feasible Schubert position for G/P . Given any feasible Schubert position $\lambda_1, \dots, \lambda_s$ for L/Q , we have the inequality*

$$(7) \quad \sum_{i=1}^s \text{codim } \varphi_{\mathfrak{s}}(T_{\pi_i \lambda_i}) \leq \dim \mathfrak{s}.$$

We prove Theorem 2 in Section 2.3.

Remark 3. Note that each inequality (7) is a combinatorial condition: As $T_{\pi_i \lambda_i}$ is H -invariant, the left hand side can be calculated explicitly using

$$\text{codim } \varphi_{\mathfrak{s}}(T_{\pi_i \lambda_i}) = |\Phi(\mathfrak{s}^*) - \Phi(T_{\pi_i \lambda_i})| = |\Phi(\mathfrak{s}^*) - \text{Inv}(\pi_i \lambda_i)|.$$

As Q , \mathfrak{s} , and $\lambda_1, \dots, \lambda_s$ range over all possibilities, this gives a system of necessary inequalities for the feasible Schubert position π_1, \dots, π_s .

The inequalities of Theorem 2 are more general than those given in the introduction. They specialize to a number of previously known inequalities, which we discuss further in Section 5. For our main theorem, we specialize to the case where $\mathfrak{s} = Z(\mathfrak{n}_R)$, the center of the nilradical of \mathfrak{r} . In this case we write φ_R for $\varphi_{Z(\mathfrak{n}_R)}$. Then the inequality (7) becomes

$$(8) \quad \sum_{i=1}^s \operatorname{codim} \varphi_R(T_{\pi_i \lambda_i}) \leq \dim Z(\mathfrak{n}_R).$$

Suppose G/P is a cominuscule flag variety. Let $M(P)$ be the set of standard parabolic subgroups of L which are equal to the stabilizer of the tangent space (at some point) to some L -orbit on $\mathfrak{g}/\mathfrak{p}$. We will show (Lemma A.7) that if $Q \in M(P)$, then L/Q is cominuscule; however, not all parabolic subgroups Q of L with L/Q cominuscule are conjugate to a subgroup in $M(P)$ (see Sections 4.1 and 4.5).

We now state our main theorem, which is proved in Section 3.

Theorem 4. *Suppose that the semisimple part of G is simple (see Remark 6). Let π_1, \dots, π_s be a Schubert position for a cominuscule flag variety G/P . Then π_1, \dots, π_s is feasible if and only if the following condition holds: for every $Q \in M(P) \cup \{L\}$ and every feasible top-degree Schubert position $\lambda_1, \dots, \lambda_s$ for L/Q , the inequality (8) holds.*

The degenerate case of $Q = L$ in (8) gives the *basic codimension inequality*

$$(9) \quad \sum \operatorname{codim} T_{\pi_i} \leq \dim G/P.$$

If we restrict our attention to top-degree Schubert positions π_1, \dots, π_s , this degenerate case is unneeded as (9) is then an equality. Thus we have the following recursion purely for the feasible top-degree Schubert positions.

Corollary 5. *Suppose that the semisimple part of G is simple. Let π_1, \dots, π_s be a top-degree Schubert position for a cominuscule flag variety G/P . Then π_1, \dots, π_s is feasible if and only if for every $Q \in M(P)$ and every feasible top-degree Schubert position $\lambda_1, \dots, \lambda_s$ for L/Q , the inequality (8) holds.*

Remark 6. The hypothesis that G^{ss} be simple is technically necessary, but mild. Theorem 4 and Corollary 5 allow us to obtain necessary and sufficient inequalities for any reductive group G and cominuscule G/P . When G^{ss} is not simple, the group $G/Z(G)$ is the product $G^1 \times \dots \times G^k$ of simple groups, and $P/Z(G)$ is the product $P^1 \times \dots \times P^k$ of parabolic subgroups $P^j \subset G^j$. Then $G/P \cong G^1/P^1 \times \dots \times G^k/P^k$, where each G^j/P^j is cominuscule (or $P^j = G^j$). Furthermore, each Schubert position $\pi_i \in W^P$ is a k -tuple $(\pi_i^1, \dots, \pi_i^k) \in W^{P^1} \times \dots \times W^{P^k}$, and π_1, \dots, π_s is feasible for G/P if and only if π_1^j, \dots, π_s^j is feasible for G^j/P^j for all j . Thus we simply check that each π_1^j, \dots, π_s^j satisfies the inequalities (8) with $Q \in M(P^j) \cup \{L^j\}$ for all j .

These inequalities are not of the form (8) on G/P , but rather of the more general form (7) on G/P .

Remark 7. In [2], Belkale showed that the Horn recursion implies Zelevinsky's Saturation Conjecture. As we will see in Sections 4.1 and 5.1, our recursion for Grassmannians is

different from the classical Horn recursion. Nevertheless, Belkale's argument can be used to show that our recursion also implies the Saturation Conjecture. We will not repeat the argument here, but the reader who is familiar with it will see that little modification is required. Thus our proof of Theorem 4 will implicitly also give a new proof of the Saturation Conjecture.

Remark 8. As can be seen from the examples in Sections 4.2 and 4.3, the system of inequalities in Theorem 4 may be redundant. An interesting problem is to find an irredundant subset of these inequalities which solves the feasibility question. For the classical Horn inequalities, this is known [1, 17], however since our inequalities are different, this problem is open, even for the Grassmannian.

2.2. Local criteria for feasibility. The derivation of necessary inequalities of Theorem 2 begins with the observation that feasibility can be detected locally. Recall that P acts on the tangent space $T_{eP}G/P \simeq \mathfrak{g}/\mathfrak{p}$.

Proposition 9. *A Schubert position π_1, \dots, π_s for G/P is feasible if and only if the intersection*

$$(10) \quad p_1 T_{\pi_1} \cap p_2 T_{\pi_2} \cap \cdots \cap p_s T_{\pi_s}$$

is transverse in $\mathfrak{g}/\mathfrak{p}$, for general $p_1, \dots, p_s \in P$.

Proof. Since a general intersection of Schubert varieties is transverse at the generic point of each of its components, either a general intersection is empty or else it is (i) non-empty, (ii) of the expected dimension, and (iii) the Schubert varieties meet transversally at every such generic point. Thus, given an intersection (6) which is non-empty but otherwise general, *either* it is transverse at the generic point of every component and the Schubert position is feasible, *or else* it is not transverse at the generic point of some component and the Schubert position is infeasible.

Consider an intersection of Schubert varieties (6) that are general subject to their containing the distinguished point eP . Such an intersection is of the form

$$(11) \quad p_1 X_{\pi_1} \cap p_2 X_{\pi_2} \cap \cdots \cap p_s X_{\pi_s},$$

where p_1, \dots, p_s are general elements of P . Since G/P is a homogeneous space, a general intersection (11) containing eP is transverse if and only if a non-empty but otherwise general intersection (6) is transverse. But (10) is just the intersection of the tangent spaces at eP to the Schubert varieties in (11). Thus the intersection (10) is transverse if and only if π_1, \dots, π_s is feasible. \square

When G/P is a cominuscule flag variety, we have the following refinement of Proposition 9, in which the general elements $p_1, \dots, p_s \in P$ are replaced by general elements $l_1, \dots, l_s \in L$ in (10).

Proposition 10. *A Schubert position π_1, \dots, π_s for a cominuscule flag variety G/P is feasible if and only if the intersection*

$$l_1 T_{\pi_1} \cap l_2 T_{\pi_2} \cap \cdots \cap l_s T_{\pi_s}$$

is transverse for generic $l_i \in L$.

Proof. Since G/P is cominuscule, the unipotent radical N_P of P is abelian and thus acts trivially on its Lie algebra \mathfrak{n}_P and on its dual, $\mathfrak{g}/\mathfrak{p}$. Thus we may replace general elements $p_1, \dots, p_s \in P$ by general elements $l_1, \dots, l_s \in L$ in (10). \square

2.3. Derivation of necessary inequalities.

Proposition 11. *For each $i = 1, \dots, s$, let π_i and λ_i be Schubert positions for G/P and L/Q , respectively, and $\pi_i \lambda_i$ the corresponding Schubert position for G/R .*

- (i) *If $\pi_1 \lambda_1, \dots, \pi_s \lambda_s$ is feasible, then so is π_1, \dots, π_s .*
- (ii) *If both $\lambda_1, \dots, \lambda_s$ and π_1, \dots, π_s is feasible, then $\pi_1 \lambda_1, \dots, \pi_s \lambda_s$ is feasible.*

Proof. For (i), the hypotheses imply that on G/R the intersection

$$(12) \quad g_1 X_{\pi_1 \lambda_1} \cap g_2 X_{\pi_2 \lambda_2} \cap \dots \cap g_s X_{\pi_s \lambda_s}$$

is non-empty for any $g_1, \dots, g_s \in G$. Since the image in G/P of this intersection under the projection map pr (3) is a subset of

$$(13) \quad g_1 \lambda_1^{-1} X_{\pi_1} \cap g_2 \lambda_2^{-1} X_{\pi_2} \cap \dots \cap g_s \lambda_s^{-1} X_{\pi_s},$$

this latter intersection is non-empty for any $g_1, g_2, \dots, g_s \in G$, which proves (i).

For (ii), let $p_1, \dots, p_s \in P$ be general. Then $p_1 \lambda_1^{-1}, \dots, p_s \lambda_s^{-1}$ are general, and the hypotheses imply that the intersection

$$p_1 \lambda_1^{-1} X_{\pi_1} \cap p_2 \lambda_2^{-1} X_{\pi_2} \cap \dots \cap p_s \lambda_s^{-1} X_{\pi_s}$$

is transverse at the point eP . Similarly, the hypotheses imply that the intersection in $L/Q = P/R$

$$p_1 X_{\lambda_1} \cap p_2 X_{\lambda_2} \cap \dots \cap p_s X_{\lambda_s}$$

is non-empty and transverse at the generic point of each component. Thus Proposition 1(ii) implies that the intersection

$$p_1 X_{\pi_1 \lambda_1} \cap p_2 X_{\pi_2 \lambda_2} \cap \dots \cap p_s X_{\pi_s \lambda_s}$$

is transverse at a general point lying in the fiber P/R above eP . \square

Using Proposition 11, we prove Theorem 2.

Proof of Theorem 2. By Proposition 11(ii), the Schubert position $\pi_1 \lambda_1, \dots, \pi_s \lambda_s$ is feasible for G/R . Let r_1, \dots, r_s be general elements of R . Then by Proposition 9

$$r_1 T_{\pi_1 \lambda_1} \cap r_2 T_{\pi_2 \lambda_2} \cap \dots \cap r_s T_{\pi_s \lambda_s}$$

is transverse.

Since $\varphi_{\mathfrak{s}}$ is a surjection, Proposition 1(i) implies that the intersection

$$(14) \quad \varphi_{\mathfrak{s}}(r_1 T_{\pi_1 \lambda_1}) \cap \varphi_{\mathfrak{s}}(r_2 T_{\pi_2 \lambda_2}) \cap \dots \cap \varphi_{\mathfrak{s}}(r_s T_{\pi_s \lambda_s})$$

is transverse in \mathfrak{s}^* . This implies the codimension inequality

$$\sum_{i=1}^s \text{codim } \varphi_{\mathfrak{s}}(r_i T_{\pi_i \lambda_i}) \leq \dim \mathfrak{s}.$$

Since the map $\varphi_{\mathfrak{s}}$ is R -equivariant, these codimensions do not depend upon the choices of the r_i , which proves the theorem. \square

3. PROOF OF THEOREM 4

This proof is independent of Lie type and uses some technical results involving roots of the different groups (G, P, R, L, Q, \dots) and their Lie algebras, which we have collected together in the Appendix. For the classical groups, these results can also be verified directly. For example, Lemma A.7 shows that L/Q is cominuscule if $Q \in M(P)$; this is also seen more concretely in Section 4 on a case-by-case basis. Figures 1, 2, and 3 illustrate the various groups and spaces that arise through examples in type A . In this case, $G/P = \text{Gr}(k, n)$, the Grassmannian of k -planes in \mathbb{C}^n , the semisimple part of L is $SL_k \times SL_{n-k}$, and the tangent space at eP is identified with $k \times (n-k)$ matrices.

We will prove Theorem 4 in three stages, which we formulate below.

Theorem 12. *Suppose that the semisimple part of G is simple. Let π_1, \dots, π_s be a Schubert position for a cominuscule flag variety G/P . Then π_1, \dots, π_s is feasible if and only if any of the following equivalent conditions hold.*

- (i) *For every $Q \in M(P) \cup \{L\}$ and every feasible Schubert position $\lambda_1, \dots, \lambda_s$ for L/Q , the intersection*

$$(15) \quad \varphi_R(r_1 T_{\pi_1 \lambda_1}) \cap \varphi_R(r_2 T_{\pi_2 \lambda_2}) \cap \dots \cap \varphi_R(r_s T_{\pi_s \lambda_s}).$$

is transverse for general elements $r_1, \dots, r_s \in R$.

- (ii) *For every $Q \in M(P) \cup \{L\}$ and every feasible Schubert position $\lambda_1, \dots, \lambda_s$ for L/Q , the inequality (8) holds.*

- (iii) *For every $Q \in M(P) \cup \{L\}$ and every feasible top-degree Schubert position $\lambda_1, \dots, \lambda_s$ for L/Q , the inequality (8) holds.*

The intersection (15) is the specialization of (14) to the case where $\mathfrak{s} = Z(\mathfrak{n}_R)$, and so the transversality of this intersection implies the inequality (8). Thus the purely combinatorial statement of (ii) above is *a priori* strictly stronger than (i), while (iii) is strictly stronger than (ii). Theorem 12(iii) is precisely Theorem 4.

Suppose that π_1, \dots, π_s is a Schubert position for G/P and l_1, \dots, l_s are general elements of L . By Proposition 10, π_1, \dots, π_s is feasible if and only if the intersection

$$(16) \quad T := l_1 T_{\pi_1} \cap l_2 T_{\pi_2} \cap \dots \cap l_s T_{\pi_s}$$

is transverse.

Since Theorem 2 establishes one direction of Theorem 12, we assume that the Schubert position π_1, \dots, π_s is infeasible, and hence that the intersection (16) is non-transverse when l_1, \dots, l_s are general elements of L . We first show that there is some $Q \in M(P)$ and a feasible Schubert position $\lambda_1, \dots, \lambda_s$ for L/Q such that a general intersection (15) is non-transverse. This will prove Theorem 12(i). Then, we use an inductive argument to show this implies that one of the inequalities (8) is violated.

3.1. A lemma on tangent spaces. Since L has only finitely many orbits on the tangent space $\mathfrak{g}/\mathfrak{p}$, there is a unique largest orbit O meeting the intersection T . This orbit does not depend on the generically chosen l_1, \dots, l_s . Set $V_i := (T_{\pi_i} \cap O)_{\text{red}}$ to be the variety underlying the scheme-theoretic intersection of T_{π_i} with this orbit.

For any $v \in \mathfrak{g}/\mathfrak{p}$, we consider its L -orbit, $L \cdot v$. As group schemes over \mathbb{C} are reduced, the tangent space to $L \cdot v$ at v is $\mathfrak{l} \cdot v$. Let \mathfrak{z} be the quotient of $\mathfrak{g}/\mathfrak{p}$ by its subspace $\mathfrak{l} \cdot v$, and let $\psi: \mathfrak{g}/\mathfrak{p} \twoheadrightarrow \mathfrak{z}$ be the quotient map.

The main idea in our proof is the following result concerning the images of the subspaces $l_i T_{\pi_i}$ in \mathfrak{z} .

Lemma 13. *Assume either that v is a general point of $T \cap O$, or that v is a smooth point of each of the varieties $l_i V_i$. The intersection (16) is transverse if and only if the intersection*

$$(17) \quad \psi(l_1 T_{\pi_1}) \cap \psi(l_2 T_{\pi_2}) \cap \cdots \cap \psi(l_s T_{\pi_s})$$

is transverse in the quotient space \mathfrak{z} .

Lemma 13 is invoked twice; once when v is taken to be a general point of $T \cap O$, and a second time when the varieties $l_i V_i$ are smooth at v (but v is chosen in advance, so *a priori* we do not know that it is sufficiently general). A consequence of our analysis is that smoothness of the $l_i V_i$ at v is the condition for v to be general.

We note that the intersection (16) is transverse if and only if for any $k \in L$, the intersection

$$kT = (kl_1)T_{\pi_1} \cap (kl_2)T_{\pi_2} \cap \cdots \cap (kl_s)T_{\pi_s}$$

is transverse. When necessary we will therefore allow ourselves to translate T , and hence v , by an element of L .

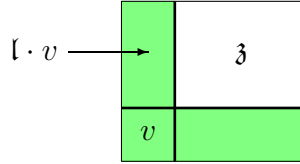


FIGURE 1. For the Grassmannian $\text{Gr}(k, n)$, we may assume that v is a rank r matrix concentrated in the lower left of $T_{eP}\text{Gr}(k, n) = \text{Mat}_{k \times (n-k)}$. Then $\mathfrak{l} \cdot v$ (shaded) and \mathfrak{z} are as shown.

Remark 14. Two special cases are worthy of immediate notice.

Suppose that v lies in the dense orbit of L . Then \mathfrak{z} is zero-dimensional, and so Lemma 13 implies that the intersection (16) is necessarily transverse.

On the other hand, suppose that $v = 0$. Then Lemma 13 provides no information. However, since v is assumed to lie in the largest orbit meeting T , we deduce that the subspaces $l_i T_{\pi_i}$ meet only at the origin, and so $\sum \text{codim } T_{\pi_i} \geq \dim \mathfrak{g}/\mathfrak{p}$. Thus the intersection (16) is transverse only when this is an equality.

Since we assumed that the intersection (16) is non-transverse, we deduce that v cannot lie in the dense orbit. Moreover, if $v = 0$, then the basic codimension inequality (9) arising from the degenerate case $Q = L$ is violated. This second observation will form the base case of the induction in our proof of Theorem 12(ii). Thus once we have proved Lemma 13 we will assume that $v \neq 0$, and that v does not lie in the dense orbit of L on $\mathfrak{g}/\mathfrak{p}$, as we have already dealt with these cases.

3.2. Proof of Lemma 13. Under either hypothesis, we have $v \in O$, hence $O = L \cdot v$. For each $i = 1, \dots, s$, we consider the scheme-theoretic intersection $l_i T_{\pi_i} \cap O$, whose underlying variety is $l_i V_i$. Let S_i denote the Zariski tangent space at v to this scheme.

$$S_i := T_v(l_i T_{\pi_i} \cap (L \cdot v)) = l_i T_{\pi_i} \cap (\mathfrak{l} \cdot v)$$

Then $S_i \supset T_v(l_i V_i)$.

Lemma 15. *Under the hypotheses of Lemma 13, the varieties $l_i V_i$ intersect transversally at v in O . Hence, the linear spaces $T_v(l_i V_i)$ are transverse in $\mathfrak{l} \cdot v$.*

Proof. Since $T \cap O$ is non-empty for generally chosen l_1, \dots, l_s , the intersection of general L -translates of the varieties V_i can never be empty. Since O is a homogeneous space of a reductive group, Kleiman's Transversality Theorem [14, Theorem 2(ii)] implies that the intersection of general L -translates of the V_i is transverse. The point v lies in the intersection of the varieties $l_i V_i$. Since the elements $l_i \in L$ were chosen to be general, we conclude that the varieties $l_i V_i$ meet transversally at v , which by (either of) the hypotheses of Lemma 13 is a general point of their intersection. \square

Corollary 16. *The linear subspaces S_i are transverse in $\mathfrak{l} \cdot v$.*

Lemma 13 now follows from Proposition 1(ii): we have the exact sequence

$$0 \longrightarrow \mathfrak{l} \cdot v \longrightarrow \mathfrak{g}/\mathfrak{p} \xrightarrow{\psi} \mathfrak{z} \longrightarrow 0.$$

with subspaces $l_i T_{\pi_i} \subset \mathfrak{g}/\mathfrak{p}$, and $S_i = l_i T_{\pi_i} \cap (\mathfrak{l} \cdot v)$ are transverse in $\mathfrak{l} \cdot v$. \square

3.3. Proof of Theorem 12(i). We now show that Lemma 13 implies Theorem 12(i) by identifying the intersection (17) in \mathfrak{z} with a general intersection of the form (15) in $Z(\mathfrak{n}_R)^*$, for a parabolic subgroup R of P corresponding to some $Q \in M(P)$.

To this end, let v be a general point of $T \cap O$, and let $Q \subset L$ be the stabilizer of $\mathfrak{l} \cdot v$. By Lemma A.7, Q is a parabolic subgroup of L and L/Q is a cominuscule flag variety. Translating v by an element of L , we may furthermore assume that Q is a standard parabolic, i.e. that $Q \supset B_L$.

Define λ_i to be the Schubert position of $l_i^{-1} Q l_i$ with respect to B_L . Then there exists a $b_i \in B_L$ such that $b_i^{-1} l_i^{-1} Q l_i b_i \supset \lambda_i B_L \lambda_i^{-1}$. Set $q_i := l_i b_i \lambda_i \in Q$. Note that $\lambda_1, \dots, \lambda_s$ is automatically feasible, since the l_i are generic and eQ lies in the intersection of the translated Schubert cells $l_i B_L \lambda_i Q$.

By Corollary A.9 we have an R -equivariant isomorphism $Z(\mathfrak{n}_R)^* \simeq \mathfrak{z}$.

Lemma 17. *We have $\varphi_R(q_i T_{\pi_i \lambda_i}) \simeq \psi(l_i T_{\pi_i})$.*

Proof. Note that $B_L \subset \pi_i^{-1} B \pi_i$. Since $B_L \subset P$, it stabilizes both P and X_{π_i} , and thus it stabilizes T_{π_i} . We have the exact sequence (5) from Section 1.2,

$$T_{\lambda_i} \hookrightarrow T_{\pi_i \lambda_i} \twoheadrightarrow \lambda_i^{-1} T_{\pi_i}.$$

Since Q stabilizes the tangent spaces $\mathfrak{l}/\mathfrak{q}$, $\mathfrak{g}/\mathfrak{r}$, and $\mathfrak{g}/\mathfrak{p}$, we may act on this sequence by $q_i := l_i b_i \lambda_i$ to obtain

$$q_i T_{\lambda_i} \hookrightarrow q_i T_{\pi_i \lambda_i} \twoheadrightarrow l_i T_{\pi_i},$$

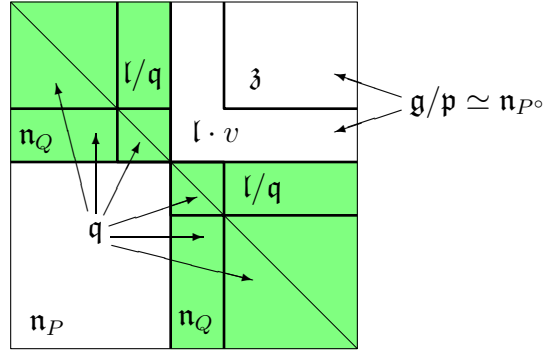


FIGURE 2. For the Grassmannian $\text{Gr}(k, n)$, if v has rank r , then $L/Q \simeq \text{Gr}(k-r, n) \times \text{Gr}(n-k-r, n-k)$. We illustrate the weight decomposition of spaces $\mathfrak{g}/\mathfrak{p}$, $\mathfrak{l} \cdot v$, \mathfrak{z} , \mathfrak{q} , $\mathfrak{l}/\mathfrak{q}$, \mathfrak{n}_P , and \mathfrak{n}_Q , where the off-diagonal entries in an $n \times n$ matrix represent the roots in A_{n-1} . The roots of \mathfrak{l} are shaded.

as $b_i \in B_L$ stabilizes T_π . This is a subdiagram of

$$\begin{array}{ccccc} \mathfrak{l}/\mathfrak{q} & \hookrightarrow & \mathfrak{g}/\mathfrak{r} & \twoheadrightarrow & \mathfrak{g}/\mathfrak{p} \\ & & \varphi_R \downarrow & & \downarrow \psi \\ & & Z(\mathfrak{n})^* & \xrightarrow{\sim} & \mathfrak{z} \end{array}$$

We conclude that $\varphi_R(q_i T_{\pi_i \lambda_i}) \simeq \psi(l_i T_{\pi_i})$, under the identification of $Z(\mathfrak{n}_R)^*$ with \mathfrak{z} . \square

Since the intersection (16) is assumed to be non-transverse, Lemma 13 implies that the intersection (17) is non-transverse. Lemma 17 shows that this is equivalent to

$$\varphi_R(q_1 T_{\pi_1 \lambda_1}) \cap \varphi_R(q_2 T_{\pi_2 \lambda_2}) \cap \cdots \cap \varphi_R(q_s T_{\pi_s \lambda_s})$$

being non-transverse.

This is an intersection of the form (15), however, since the q_i are constructed from v and l_i , they will not be general elements of R (they are not even general elements of Q). It remains to show that a *general* intersection (15) is non-transverse.

Consider what happens when we translate each l_i by a general element $k_i \in \text{Stab}_L(\mathbb{C}v) \subset Q$. The point v will still be a point of the new intersection

$$T' := (k_1 l_1) T_{\pi_1} \cap (k_2 l_2) T_{\pi_2} \cap \cdots \cap (k_s l_s) T_{\pi_s},$$

thus we obtain the same subgroup Q . Moreover, since v is a smooth point of $l_i V_i$, and the k_i are general, it will be a smooth point of $(k_i l_i) V_i$. If q'_i denotes the new q_i we obtain for the intersection T' , we find that $q'_i = k_i q_i$. Thus by Lemmas 13 and 17 we see that the intersection

$$\varphi_R(k_1 q_1 T_{\pi_1 \lambda_1}) \cap \varphi_R(k_2 q_2 T_{\pi_2 \lambda_2}) \cap \cdots \cap \varphi_R(k_s q_s T_{\pi_s \lambda_s})$$

is non-transverse for general $k_i \in \text{Stab}_L(\mathbb{C}v)$. By Lemma A.14, this implies that a general intersection (15) is non-transverse. This proves Theorem 12(i). \square

3.4. Proof of Theorem 12(ii). Recall that $M(P)$ is exactly the set of those standard parabolic subgroups of the form $\text{Stab}_L(\mathfrak{l} \cdot v)$ for some $v \in \mathfrak{g}/\mathfrak{p}$.

We show that if π_1, \dots, π_s is an infeasible Schubert position for G/P , then there is a parabolic subgroup $Q \in M(P)$ of L and a feasible Schubert position $\lambda_1, \dots, \lambda_s$ for L/Q such that

$$(18) \quad \sum_{i=1}^s \text{codim } \varphi_R(T_{\pi_i \lambda_i}) > \dim Z(\mathfrak{n}_R),$$

where R is the parabolic subgroup of P containing Q .

Suppose that Theorem 12(ii) holds for any proper subgroup of G whose semisimple part is simple, and let π_1, \dots, π_s be an infeasible Schubert position for G/P . By Theorem 12(i), there is a parabolic subgroup $Q \in M(P)$ and a feasible Schubert position $\lambda_1, \dots, \lambda_s$ for L/Q such that for general $r_1, \dots, r_s \in R$, the intersection

$$(19) \quad \varphi_R(r_1 T_{\pi_1 \lambda_1}) \cap \varphi_R(r_2 T_{\pi_2 \lambda_2}) \cap \dots \cap \varphi_R(r_s T_{\pi_s \lambda_s})$$

is not transverse. If this intersection has dimension 0, then we deduce the codimension inequality (18) and so we are done.

Now we assume that the dimension of the intersection (19) is not zero, and we use our inductive hypothesis to find a different parabolic subgroup $Q_1 \in M(P)$ and a feasible Schubert position μ_1, \dots, μ_s for L/Q_1 so that the corresponding inequality holds.

We begin by constructing a new cominuscle flag variety G'/P' whose tangent space at eP' is identified with \mathfrak{z} . This will allow us to identify the intersection (19) as an intersection of tangent spaces of Schubert varieties. Define the reductive (proper) subgroup G' of G to be

$$G' := Z_G(Z_H(Z(N_R))).$$

G' is the smallest reductive subgroup of G containing both H and $Z(N_R)$. Set $P' := G' \cap R$. Let L' denote the Levi subgroup of P' , and let W' denote the Weyl group of G' .

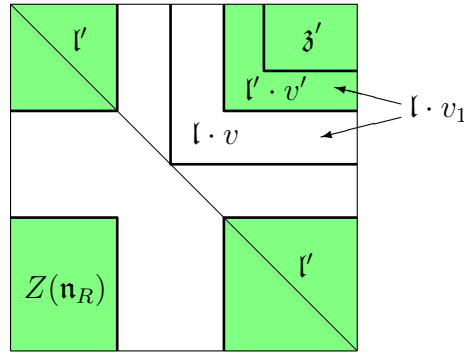


FIGURE 3. For the Grassmannian $\text{Gr}(k, n)$, the semisimple part of G' is isomorphic to $SL_{(k-r)+(n-k-r)} = SL_{n-2r}$, whose roots are shaded. We also illustrate the weights of \mathfrak{l}' , $\mathfrak{l} \cdot v$, $\mathfrak{l}' \cdot v'$, \mathfrak{z}' , $\mathfrak{l} \cdot v_1$, and $Z(\mathfrak{n}_R)$.

By Lemma A.12 the semisimple part of G' is simple and by Lemma A.13 G'/P' is cominuscle. Thus the inductive hypothesis applies to G'/P' .

The pattern map $w \mapsto \bar{w}$ of Billey and Braden [4] sends $W \rightarrow W'$. The element $\bar{w} \in W'$ is defined by its inversion set, which is $\Phi(\mathfrak{g}') \cap \text{Inv}(w)$.

Lemma 18. *For all $w \in W^R$, $\varphi_R(T_w) = T_{\bar{w}}$.*

Proof. Since $w \in W^R$, $\text{Inv}(w) = \Phi(T_w)$. The weights of the tangent space $T_{\bar{w}}$ are the inversions of \bar{w} which lie in $\Phi(\mathfrak{g}'/\mathfrak{p}')$. By Lemma A.13, $\Phi(\mathfrak{g}'/\mathfrak{p}') = \Phi(\mathfrak{z})$. Since the weights of $\varphi_R(T_w)$ are $\text{Inv}(w) \cap \Phi(\mathfrak{z})$, we are done. \square

By Lemmas 18 and A.14 there exist $l'_1, \dots, l'_s \in L'$ such that the intersection (19) is equal to

$$l'_1 T_{\pi_1 \lambda_1} \cap l'_2 T_{\pi_2 \lambda_2} \cap \dots \cap l'_s T_{\pi_s \lambda_s}.$$

Furthermore, as the elements $r_i \in R$ are general, so are the elements $l'_i \in L'$. Since this intersection is not transverse, we conclude that if we set $\pi'_i := \overline{\pi_i \lambda_i}$, then π'_1, \dots, π'_s is an infeasible Schubert position for G'/P' .

By our inductive hypothesis, there is a parabolic subgroup $Q' \in M(P')$ and feasible Schubert positions $\lambda'_1, \dots, \lambda'_s$ such that

$$(20) \quad \sum_{i=1}^s \text{codim } \varphi'_{R'}(T_{\pi'_i \lambda'_i}) > \dim Z(\mathfrak{n}_{R'}).$$

(Here, $R' \subset P'$ is the largest parabolic subgroup such that $R' \cap L' = Q'$.) Then Q' is a standard parabolic which stabilizes $\mathfrak{l}' \cdot v'$ for some $v' \in \mathfrak{g}'/\mathfrak{p}' (\simeq \mathfrak{z})$.

Let Q_1 be the stabilizer in L of $\mathfrak{l} \cdot v_1$, where $v_1 = v + v'$ (we consider v' to be an element of $\mathfrak{g}/\mathfrak{p}$ by the L' -equivariant injection $\mathfrak{g}'/\mathfrak{p}' \hookrightarrow \mathfrak{g}/\mathfrak{p}$). It follows from Lemma A.15 that Q_1 is a standard parabolic, and so $Q_1 \in M(P)$. Let R_1 be the corresponding parabolic subgroup of P . By Lemma A.16, $Z(\mathfrak{n}_{R_1}) = Z(\mathfrak{n}_{R'})$, and $\mathfrak{z}' = (\mathfrak{g}'/\mathfrak{p}')/(\mathfrak{l}' \cdot v')$ is the dual to this space.

Let μ_i be the minimal coset representative of $\lambda_i \lambda'_i$ in W_L/W_{Q_1} . Since $\lambda_1, \dots, \lambda_s$ is feasible for $L/Q = P/R$, and $\lambda'_1, \dots, \lambda'_s$ is feasible for $L'/Q' = R/(R \cap R_1)$, $\lambda_1 \lambda'_1, \dots, \lambda_s \lambda'_s$ is feasible for $P/(R \cap R_1)$, by Proposition 11(ii). Hence by Proposition 11(i), μ_1, \dots, μ_s is feasible for $P/R_1 = L/Q_1$.

We now complete the proof by showing that $\dim \varphi_{R_1}(T_{\pi_i \mu_i}) = \dim \varphi'_{R'}(T_{\pi'_i \lambda'_i})$. These H -invariant subspaces have weights $\text{Inv}(\pi_i \mu_i) \cap \Phi(\mathfrak{z}')$ and $\text{Inv}(\overline{\pi_i \lambda_i \lambda'_i}) \cap \Phi(\mathfrak{z}')$, respectively. Let $\nu_i = \mu_i^{-1} \lambda_i \lambda'_i \in W_{Q_1}$. Then by (4),

$$\text{Inv}(\pi_i \mu_i) \cap \Phi(\mathfrak{z}') = (\nu_i \text{Inv}(\pi_i \lambda_i \lambda'_i)) \cap \Phi(\mathfrak{z}') = \nu_i (\text{Inv}(\pi_i \lambda_i \lambda'_i) \cap \Phi(\mathfrak{z}')),$$

as W_{Q_1} preserves $\Phi(\mathfrak{z}')$. Thus it suffices to show that

$$\text{Inv}(\pi_i \lambda_i \lambda'_i) \cap \Phi(\mathfrak{z}') = \text{Inv}(\overline{\pi_i \lambda_i \lambda'_i}) \cap \Phi(\mathfrak{z}').$$

Note that we have $\overline{\pi \lambda \lambda'} = \overline{\pi} \overline{\lambda} \lambda'$, as the pattern map is W' -equivariant. Then indeed

$$\begin{aligned} \text{Inv}(\pi \lambda \lambda') \cap \Phi(\mathfrak{z}') &= (\pi \lambda \lambda')^{-1} \Phi^- \cap \Phi^+(\mathfrak{g}') \cap \Phi(\mathfrak{z}') \\ &= \text{Inv}(\overline{\pi \lambda \lambda'}) \cap \Phi(\mathfrak{z}') \\ &= \text{Inv}(\overline{\pi} \overline{\lambda} \lambda') \cap \Phi(\mathfrak{z}'). \end{aligned}$$

Thus we have exhibited a parabolic subgroup $Q_1 \in M(P)$ and a feasible Schubert position μ_1, \dots, μ_s for L/Q_1 , such that by rewriting (20) we have

$$\sum_{i=1}^s \operatorname{codim} \varphi_{R_1}(T_{\pi_i \mu_i}) > \dim Z(\mathfrak{n}_{R_1}),$$

as required. \square

3.5. Proof of Theorem 12(iii). We need the following non-obvious fact which is proven in the Ph.D. Thesis [21].

Proposition 19. *Suppose that $\pi' < \pi$ in the Bruhat order. Then there is an injection $\iota: \operatorname{Inv}(\pi') \hookrightarrow \operatorname{Inv}(\pi)$ such that if $\alpha \in \operatorname{Inv}(\pi')$, then $\iota(\alpha)$ is a higher root than α .*

Sketch of Proof. It is enough to show this when π' covers π in the Bruhat order. In this case, π' and π differ by reflection in a root β , and one can verify the proposition by comparing inversions within strings of roots along lines parallel to β . \square

Let π_1, \dots, π_s be an infeasible Schubert position for G/P . Then by Theorem 12(ii), there exists a parabolic subgroup $Q \in M(L)$ and a feasible Schubert position $\lambda_1, \dots, \lambda_s$ for L/Q such that the inequality (18) holds.

If this Schubert position for L/Q does not have top-degree, then by Chevalley's formula [6], there exists a feasible Schubert position μ_1, \dots, μ_s for L/Q such that $\mu_i \leq \lambda_i$, for $i = 1, \dots, s$. Since each π_i is a minimal coset representative, we have $\pi_i \mu_i \leq \pi_i \lambda_i$. Recall that the dimension of $\varphi_R(T_{\pi_i \lambda_i})$ is the number of inversions of $\pi_i \lambda_i$ which lie in the set of weights $\Phi(\mathfrak{z})$. Since N_R is B -stable, so is its center $Z(N_R)$, and hence the roots in $\Phi(\mathfrak{z}) = -\Phi(Z(\mathfrak{n}_R))$ are an upper order ideal in $\Phi(\mathfrak{g})$. Then Proposition 19 implies that $\dim \varphi_R(T_{\pi_i \mu_i}) \leq \dim \varphi_R(T_{\pi_i \lambda_i})$, and thus (18) holds for μ_1, \dots, μ_s in place of $\lambda_1, \dots, \lambda_s$. \square

4. EXPLICATING THE HORN RECURSION

By Theorem 4, the feasibility of a Schubert position π_1, \dots, π_s for cominusule G/P is detected by the inequality (8) for every feasible top-degree Schubert position $\lambda_1, \dots, \lambda_s$ for L/Q for every $Q \in M(P)$. We noted in Remark 3 that these inequalities are combinatorial conditions. We now reformulate this. Write $\operatorname{Inv}^c(\pi)$ for the set of weights $\Phi(\mathfrak{g}/\mathfrak{p}) - \operatorname{Inv}(\pi)$ and call these the *coinversions* of π . They are the weights of the normal bundle, $(\mathfrak{g}/\mathfrak{p})/T_\pi$, to X_π at eP .

Lemma 20. *Given a Schubert position π_1, \dots, π_s for G/P and a feasible Schubert position $\lambda_1, \dots, \lambda_s$ for L/Q with $Q \in M(P)$, the inequality (8) is equivalent to*

$$(21) \quad \sum_{i=1}^s |\operatorname{Inv}^c(\pi_i) \cap \lambda_i \Phi(\mathfrak{z})| \leq \dim \mathfrak{z},$$

where $\mathfrak{z} = Z(\mathfrak{n}_R)^*$.

Proof. As we observed in Remark 3, the inequality (7) (and hence (8)) can be computed combinatorially as $\text{codim } \varphi_{\mathfrak{s}}(T_{\pi_i \lambda_i}) = |\Phi(\mathfrak{s}^*) - \text{Inv}(\pi_i \lambda_i)|$. Since $\mathfrak{s}^* = \mathfrak{z}$, by (4) we have

$$\begin{aligned} \text{codim } \varphi_R(T_{\pi_i \lambda_i}) &= |\Phi(\mathfrak{z}) - \text{Inv}(\pi_i \lambda_i)| \\ &= |\Phi(\mathfrak{z}) \cap (\Phi(\mathfrak{g}/\mathfrak{p}) - \text{Inv}(\pi_i \lambda_i))| \\ &= |\Phi(\mathfrak{z}) - \lambda_i^{-1} \text{Inv}(\pi_i)| \\ &= |\Phi(\mathfrak{z}) \cap \lambda_i^{-1} \text{Inv}^c(\pi_i)|. \end{aligned}$$

Translating by λ_i , this is equal to $|\lambda_i \Phi(\mathfrak{z}) \cap \text{Inv}^c(\pi_i)|$, which implies the lemma. \square

We introduce the following notation. Given a Schubert position π for G/P and a Schubert position λ for L/Q , set $|\pi|_{\lambda} := |\text{Inv}^c(\pi) \cap \lambda \Phi(\mathfrak{z})|$. We also write $|\pi| := |\text{Inv}^c(\pi)| = \text{codim } T_{\pi}$. Then the inequalities of Lemma 20 become

$$\sum_{i=1}^s |\pi_i|_{\lambda_i} \leq \dim \mathfrak{z}.$$

whereas the basic codimension inequality (9) becomes

$$\sum_{i=1}^s |\pi_i| \leq \dim \mathfrak{g}/\mathfrak{p}.$$

Since G/P is cominuscule, the weights $\Phi(\mathfrak{g}/\mathfrak{p})$ form a lattice [19]. For $\pi \in W^P$, the tangent space T_{π} is B_L -invariant, so its weights form a lower order ideal in this lattice. Given a poset Y , let $J(Y)$ be the distributive lattice of lower order ideals of Y [26]. Proctor [19] showed that

Proposition 21. $W^P \simeq J(\Phi(\mathfrak{g}/\mathfrak{p}))$.

Remark 22. Proposition 21 allows us to interpret the inequalities (8) in terms of convex geometry. Let V be the vector space of functions $f: \Phi(\mathfrak{g}/\mathfrak{p}) \rightarrow \mathbb{R}$. The set

$$\mathcal{O}_{\mathfrak{g}/\mathfrak{p}} := \{f \in V \mid \alpha < \beta \in \Phi(\mathfrak{g}/\mathfrak{p}) \Rightarrow 0 \leq f(\alpha) \leq f(\beta) \leq 1\}$$

of order preserving maps from $\Phi(\mathfrak{g}/\mathfrak{p})$ to $[0, 1]$ is the order polytope [25] of the poset $\Phi(\mathfrak{g}/\mathfrak{p})$. Its integer points are the indicator functions of upper order ideals in $\Phi(\mathfrak{g}/\mathfrak{p})$, which by Proposition 21 are the indicator functions of the coinversion sets $\text{Inv}^c(\pi)$ of Schubert positions π for G/P . Write $u_{\pi} \in V$ for the integer point of $\mathcal{O}_{\mathfrak{g}/\mathfrak{p}}$ corresponding to the Schubert position π .

Given a Schubert position λ for L/Q with $Q \in M(P)$, define a linear map $\Sigma_{\lambda}: V \rightarrow \mathbb{R}$ by

$$\Sigma_{\lambda}(f) := \sum_{\gamma \in \lambda \Phi(\mathfrak{z})} f(\gamma).$$

Then $|\pi|_{\lambda} = \Sigma_{\lambda}(u_{\pi})$.

In particular, the inequality (8) may be interpreted as a linear inequality on the polytope $(\mathcal{O}_{\mathfrak{g}/\mathfrak{p}})^s$, and so the set of all feasible Schubert positions π_1, \dots, π_s for G/P is naturally identified with the integer points in the *feasibility polytope* which is the subpolytope of $(\mathcal{O}_{\mathfrak{g}/\mathfrak{p}})^s$ defined by the set of inequalities from Theorem 4. We have not studied the structure of this feasibility polytope.

We now investigate the inequalities of Theorem 4 on a case-by-case basis. Recall that $M(P)$ is the set of standard parabolic subgroups of L of the form $Q = \text{Stab}_L(T_v L \cdot v)$, for some $v \in \mathfrak{g}/\mathfrak{p}$. Any two suitable choices of v in the same L -orbit give the same Q . Thus for each type, it is enough analyze one such choice of v from each L -orbit. The cases where $v = 0$ or v is in the dense orbit can be excluded, since these yield $\text{Stab}_L(T_v L \cdot v) = L$. We can always take v to be of the form

$$v = v_{\alpha_1} + \cdots + v_{\alpha_r},$$

where $v_\alpha \in \mathfrak{g}/\mathfrak{p}$ is a non-zero vector of weight α , and $\alpha_1, \dots, \alpha_r$ is a sequence of orthogonal long roots. The number r determines the L orbit of v [22]. We will also make use of Lemma A.4, which asserts that for such a choice v , the weights of \mathfrak{z} will be the weights of $\mathfrak{g}/\mathfrak{p}$ orthogonal to $\alpha_1, \dots, \alpha_r$.

4.1. Type A_{n-1} , the classical Grassmannian, $\text{Gr}(k, n)$. Suppose that P is obtained by omitting the k th node in the Dynkin diagram of A_{n-1} . Then G/P is $\text{Gr}(k, n)$, the Grassmannian of k -planes in \mathbb{C}^n . The Levi subgroup L of P has semisimple part $SL_k \times SL_{n-k}$. We identify $\mathfrak{g}/\mathfrak{p}$ with $\text{Hom}(\mathbb{C}^k, \mathbb{C}^{n-k})$, where $\mathbb{C}^k \oplus \mathbb{C}^{n-k} = \mathbb{C}^n$. Its weights are

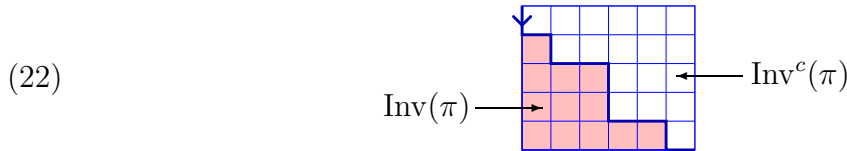
$$\Phi(\mathfrak{g}/\mathfrak{p}) = \{e_j - e_i \mid 1 \leq i \leq k < j \leq n\},$$

where e_1, \dots, e_n are the standard orthonormal basis vectors of $\mathbb{C}^n = \mathfrak{h}^*$. We identify $\Phi(\mathfrak{g}/\mathfrak{p})$ with the cells of a $k \times (n-k)$ rectangle where $e_j - e_i$ corresponds to the cell in row i (from the top) and column $j-k$ (from the left). The lowest root in $\Phi(\mathfrak{g}/\mathfrak{p})$ is in the lower left corner and the highest root is in the upper right corner.

Minimal coset representatives $\pi \in W^P$ are permutations of n with a unique descent at position k . The inversion set of a permutation π is the set of roots

$$\{e_j - e_i \mid i \leq k < j \text{ such that } \pi(i) > \pi(j)\}.$$

We display this for $n = 11$, $k = 5$, and $\pi = 1367 \, 10 \, 24589 \, 11$, shading the inversion set.



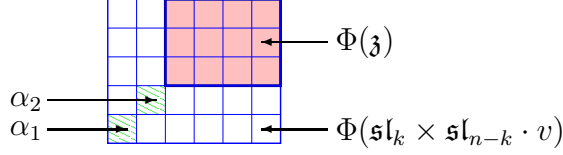
The permutation may be read off from the inversion diagram as follows. Consider the path which forms the border of $\text{Inv}(\pi)$ from the upper left corner to the lower right corner of the rectangle. If we label the steps from 1 to n , then the labels of the vertical steps are the first k values of π and the labels of the horizontal steps are the last $n-k$ values of π .

If we write $\alpha_i := e_{k+i} - e_{k+1-i}$, which is the i th root along the anti-diagonal in $\Phi(\mathfrak{g}/\mathfrak{p})$ starting from the lower left, then the vector v may be taken to have the form

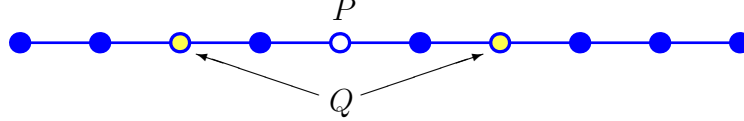
$$(23) \quad v = v_{\alpha_1} + v_{\alpha_2} + \cdots + v_{\alpha_r},$$

and $L \cdot v \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^{n-k})$ consists of rank r matrices. Note that $1 \leq r < \min\{k, n-k\}$. Then the set $\Phi(\mathfrak{z})$ is the upper right $(k-r) \times (n-k-r)$ rectangle in the $k \times (n-k)$ rectangle

representing $\Phi(\mathfrak{g}/\mathfrak{p})$, and $\dim \mathfrak{z} = (k-r)(n-k-r)$. We show this for $n = 11$, $k = 5$, and $r = 2$.

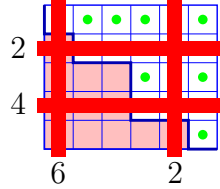


The subgroup $Q \in M(P)$ which is the stabilizer of $\mathfrak{sl}_k \times \mathfrak{sl}_{n-k} \cdot v$ is obtained by further omitting the nodes at $k-r$ and at $k+r$ in the Dynkin diagram for L^{ss} .

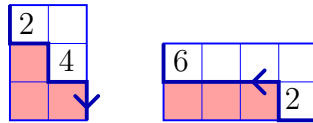


Thus L/Q is isomorphic to $\text{Gr}(k-r, k) \times \text{Gr}(r, n-k)$.

An element $\lambda \in W^Q$ acts on $\Phi(\mathfrak{g}/\mathfrak{p})$ by simultaneously shuffling the r rows that do not meet $\Phi(\mathfrak{z})$ with those that do, and the same for columns. This is equivalent to selecting r rows and r columns, the images under λ of the rows and columns which do not meet $\Phi(\mathfrak{z})$. If we draw $\text{Inv}^c(\pi)$ in the rectangle and cross out the selected rows and columns, then $|\pi|_\lambda$ is the number of boxes which remain. In the example (22) above with $\pi = 1367102458911$ and $r = 2$, if λ selects rows 2 and 4 from the top and columns 2 and 6 from the right, we see that $|\pi|_\lambda = 7$.



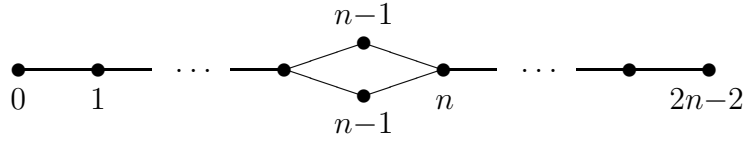
Remark 23. For the purpose of our cominuscule recursion we describe how to obtain the inversion diagram of an element $\lambda \in W^Q$, which is a subset of a $(k-r) \times r$ rectangle for the rows and a $r \times (n-k-r)$ rectangle for the columns. In the rectangle for the rows, draw a path from the upper left corner to the lower right corner whose i th step is *horizontal* if λ selected row i and *vertical* otherwise, while in the rectangle for the columns, draw a path from the lower right corner to the upper left corner whose i th step is *vertical* if λ selected column i and *horizontal* otherwise. We show this for our example.



Since L/Q is a product of smaller cominuscule flag varieties, feasibility for the Schubert positions λ in the recursion is determined separately on each factor. Note that not all cominuscule L/Q enter into this recursion.

4.2. Type D_{n+1} , $G = SO_{2n+2}$, G/P is the even-dimensional quadric, Q^{2n} . Here, the parabolic subgroup P is obtained by omitting the rightmost node of the Dynkin diagram, as shown in Table 1. Its Levi subgroup L has semisimple part SO_{2n} and the flag variety

G/P is the even-dimensional quadric Q^{2n} in \mathbb{P}^{2n+1} . The lattice $\Phi(\mathfrak{g}/\mathfrak{p})$ is the poset Λ_{n-1} , whose Hasse diagram we display, where elements to the right are greater.



Each root in $\Phi(\mathfrak{g}/\mathfrak{p})$ is orthogonal to exactly one other, and their indices sum to $2n-2$. Consequently an orthogonal sequence of long roots has length at most 2. For our purposes, there is one interesting orbit of L in $\mathfrak{g}/\mathfrak{p}$. In fact, $\mathfrak{g}/\mathfrak{p}$ is the defining representation of SO_{2n} and this orbit is the set of (non-zero) isotropic vectors, the cone over the quadric Q^{2n-2} . Thus $M(P)$ consists of a single parabolic subgroup Q , where L/Q is the quadric Q^{2n-2} , and $W^Q = \Lambda_{n-1}$. Here Q is the stabilizer in L of $\mathfrak{l} \cdot v_\alpha$, where α is the simple root defining P (labeled 0 in Λ_{n-1}) and $\Phi(\mathfrak{z})$ is the orthogonal complement to α which is the single root labelled $2n-2$ in Λ_{n-1} .

By Proposition 21, W^P is the set of order ideals of Λ_{n-1} , which is equal to Λ_n , where the set of weights of T_π is equal to the order ideal π , and $|\pi|$ is the cardinality of the *complement* of this order ideal. Thus $\lambda \in W^Q$ is an element of Λ_{n-1} , whereas $\pi \in W^P$ is an order ideal of Λ_{n-1} . The action of W_P on both $\Phi(\mathfrak{g}/\mathfrak{p})$ and W_Q canonically identifies these two occurrences of Λ_{n-1} ; however, as the identification of W^P with Λ_n is not canonical, there is a choice to be made. We will adopt the convention that $n \in \Lambda_n$ corresponds to the n -element order ideal in Λ_{n-1} which contains $n-1$ and \bar{n} corresponds to the n -element order ideal which contains $\bar{n-1}$. For $\lambda \in W^Q = \Lambda_n$, we see that $\lambda\Phi(\mathfrak{z})$ is the root λ^\perp orthogonal to λ , which is found by rotating the Hasse diagram by 180° . Thus

$$(24) \quad |\pi|_\lambda = \begin{cases} 0 & \text{if } \lambda^\perp \in \pi \\ 1 & \text{otherwise} \end{cases}$$

For example,

$$(25) \quad |n|_{\bar{n-1}} = |\bar{n}|_{n-1} = 1 \quad \text{and} \quad |n|_{n-1} = |\bar{n}|_{\bar{n-1}} = 0.$$

Since $|M(P)| = 1$ and L/Q is Q^{2n-2} , the cominuscle recursion in this case can proceed by induction on n . The base case is Q^2 , the quadric in \mathbb{P}^3 which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Note that the condition

$$\sum_{i=1}^s |\pi_i|_{\lambda_i} \leq 1,$$

for $\lambda_1, \dots, \lambda_s$ feasible for L/Q , is implied by the basic codimension inequality

$$\sum_{i=1}^s |\pi_i| \leq 2n,$$

unless $|\pi_1| + |\pi_2| = 2n$ and $|\pi_3| = \dots = 0$ (or some permutation thereof). Indeed if $\lambda_1, \dots, \lambda_s$ is feasible for L/Q , and $|\pi_1|_{\lambda_1} = |\pi_2|_{\lambda_2} = 1$, then $|\pi_1| + |\pi_2| \geq 2n$. Thus the only interesting cases are to determine which pairs (n, n) , (\bar{n}, \bar{n}) , (n, \bar{n}) are feasible.

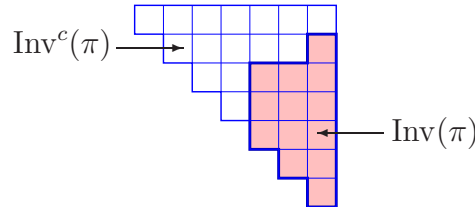
The cominuscle recursion gives this answer to this question. We use the computations (25). If $(n-1, \bar{n-1})$ is feasible for Q^{2n-2} , then $|\bar{n}|_{n-1} + |n|_{\bar{n-1}} = 2 > 1$, and so (n, \bar{n})

is infeasible for Q^{2n} , whereas (n, n) and (\bar{n}, \bar{n}) are feasible. Similarly if $(n-1, n-1)$ and $(\bar{n}-1, \bar{n}-1)$ are feasible for L/Q , then $|\bar{n}|_{n-1} + |\bar{n}|_{n-1} = |n|_{\bar{n}-1} + |n|_{\bar{n}-1} = 2 > 1$, and so (n, n) and (\bar{n}, \bar{n}) are infeasible, and (n, \bar{n}) is feasible. By induction, we see that if n is odd, (n, n) and (\bar{n}, \bar{n}) are feasible for Q^{2n} and (n, \bar{n}) is infeasible, and vice-versa if n is even.

4.3. Type B_n , $G = SO_{2n+1}$, G/P is an odd-dimensional quadric, Q^{2n-1} . The analysis of the odd-dimensional quadric is similar to the even-dimensional quadric, in that $\mathfrak{g}/\mathfrak{p}$ is the defining representation of $L = SO_{2n-1}$ and there is a single interesting L -orbit on $\mathfrak{g}/\mathfrak{p}$ consisting of non-zero isotropic vectors. In the even-dimensional quadric, this orbit gave the inequalities for determining feasibility in the middle dimension. For the odd-dimensional quadric, which has no middle-dimensional cohomology, these inequalities are redundant: they are all implied by the basic codimension inequality $\sum_{i=1}^s |\pi_i| \leq 2n - 1$. Thus feasibility for Q^{2n-1} is trivial, as the only inequality needed is the basic codimension inequality.

4.4. Type C_n , $G = Sp_{2n}$, G/P is the Lagrangian Grassmannian. Suppose that P is obtained by omitting the long root from the Dynkin diagram for C_n . Then $G/P = LG(n)$, the *Lagrangian Grassmannian* of isotropic n -planes in \mathbb{C}^{2n} , where \mathbb{C}^{2n} is equipped with a non-degenerate alternating bilinear form. The Levi subgroup of P is GL_n , and $\mathfrak{g}/\mathfrak{p}$ is the second symmetric power of the defining representation of GL_n , that is, symmetric $n \times n$ matrices. Its weights are $\{e_i + e_j \mid 1 \leq i \leq j \leq n\}$, where e_1, \dots, e_n are the standard orthonormal basis vectors of $\mathbb{C}^n = \mathfrak{h}^*$.

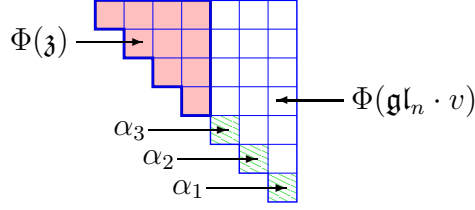
We identify $\Phi(\mathfrak{g}/\mathfrak{p})$ with the cells of the staircase shape of height n . Numbering the rows and columns in the standard way for matrices, the weight $e_i + e_j$ with $i \leq j$ corresponds to the cell in row i and column j in the staircase. We write the coinversion set of a minimal coset representative $\pi \in W^P$ as a strict partition in the staircase, with the inversion set its complement. We use the strict partition of $\text{Inv}^c(\pi)$ to represent elements $\pi \in W^P$. We display this for $n = 7$ and $\pi = 7521$, shading the inversions of π .



The lowest root in $\Phi(\mathfrak{g}/\mathfrak{p})$ is in the last row and the highest root is in the first column.

The long roots in $\Phi(\mathfrak{g}/\mathfrak{p})$ are $2e_1, \dots, 2e_n$, which are pairwise orthogonal. Set $\alpha_i := 2e_{n+1-i}$, which is the i th root along the diagonal edge of $\Phi(\mathfrak{g}/\mathfrak{p})$ from the lower right. Then the vector v has the form $v_{\alpha_1} + v_{\alpha_2} + \dots + v_{\alpha_r}$. The weights of $\mathfrak{gl}_n \cdot v$ are $\{e_i + e_j \mid n-r < j\}$, and the subgroup $Q \in M(P)$ of $L = GL_n$ which is the stabilizer of $\mathfrak{gl}_n \cdot v$ is the stabilizer of the r -dimensional linear subspace spanned by the last r basis vectors e_{n+1-r}, \dots, e_n . Thus L/Q is the classical Grassmannian, $\text{Gr}(r, n)$. In this way, the weights of $\mathfrak{gl}_n \cdot v$ are the last r columns of the staircase and the weights of \mathfrak{z} are the first $n-r$ columns and

$\dim \mathfrak{z} = \binom{n-r+1}{2}$. We show this for $n = 7$ and $r = 3$.



Elements $\lambda \in W^Q$ act on $\Phi(\mathfrak{g}/\mathfrak{p})$ by simultaneously shuffling rows and columns numbered $1, \dots, n-r$ with those numbered $n+1-r, \dots, n$. This is equivalent to selecting r boxes on the diagonal corresponding to the images of the roots $\alpha_1, \dots, \alpha_r$. Then $\lambda\Phi(\mathfrak{z})$ consists of weights which are orthogonal to each of the selected weights, and are obtained by crossing out the row and column of each selected box. This is displayed in Figure 4(a) for $n = 7$, $r = 3$, and when λ selects the boxes in positions 2, 3, and 6. After crossing out the rows

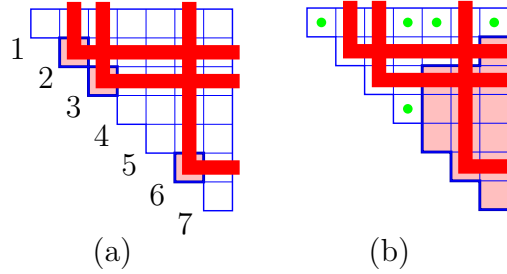
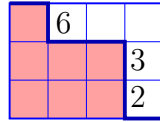


FIGURE 4. $|\pi|_\lambda = 5$ for $\pi = 7521$ and $\lambda = 236$.

and columns, $|\pi|_\lambda$ is the number of boxes in $\text{Inv}^c(\pi)$ which are not crossed out. We display this in Figure 4(b) for $\pi = 7521$ with the same numbers n , r , and λ as before.

We associate a minimal coset representative $\lambda \in W^Q$ for $\text{Gr}(r, n)$ to a selection of boxes on the diagonal in the same way as for columns in Remark 23. In our example, the selection of positions 2, 3, and 6 gives the inversion diagram for $\text{Gr}(3, 7)$.



4.5. Type D_{n+1} , $G = SO_{2n+2}$, G/P is the orthogonal Grassmannian, $OG(n+1)$. Suppose that P is obtained by omitting one of the roots in the fork of the Dynkin diagram for D_{n+1} . Then G/P is the orthogonal Grassmannian $OG(n+1)$ of isotropic $n+1$ -planes in \mathbb{C}^{2n+2} , where \mathbb{C}^{2n+2} is equipped with a non-degenerate symmetric bilinear form. The Levi subgroup of P is GL_{n+1} , and $\mathfrak{g}/\mathfrak{p}$ is the second exterior power of the defining representation of GL_{n+1} , that is, anti-symmetric $(n+1) \times (n+1)$ -matrices. Its weights are $\{e_i + e_j \mid 1 \leq i < j \leq n+1\}$.

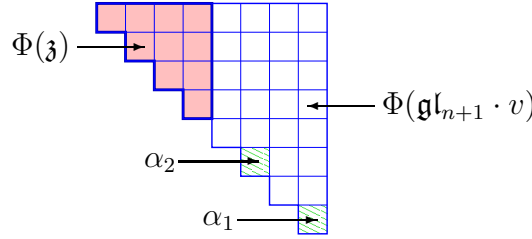
We identify $\Phi(\mathfrak{g}/\mathfrak{p})$ with the cells of the staircase shape of height n . Minimal coset representatives $\pi \in W^P$ are strict partitions corresponding to $\text{Inv}^c(\pi)$. This is exactly

the same as for the Lagrangian Grassmannian $LG(n)$; not only do these two cominuscule flag varieties have Schubert positions indexed by the same set (of strict partitions), but a Schubert position π_1, \dots, π_s is feasible for $LG(n)$ if and only if π_1, \dots, π_s is feasible for $OG(n+1)$. Despite this similarity, the minuscule recursion is different for $LG(n)$ and for $OG(n+1)$.

Numbering the rows of the staircase from 1 to n with 1 the longest row, and the columns 2 to $n+1$ with $n+1$ the longest column, the weight $e_i + e_j$ with $i < j$ corresponds to the cell in row i and column j in the staircase. Every root in $\Phi(\mathfrak{g}/\mathfrak{p})$ is long. Set $\alpha_i := e_{n+2-2i} + e_{n+3-2i}$, which is the $(2i-1)$ st root along the diagonal edge of $\Phi(\mathfrak{g}/\mathfrak{p})$ from the lower right. Then the vector v has the form

$$v_{\alpha_1} + v_{\alpha_2} + \dots + v_{\alpha_r}.$$

The weights of $\mathfrak{gl}_{n+1} \cdot v$ are $\{e_i + e_j \mid n+1-2r < j\}$, and the subgroup Q of $L = GL_{n+1}$ which stabilizes $\mathfrak{gl}_{n+1} \cdot v$ is the subgroup stabilizing the $2r$ -dimensional linear subspace spanned by the last $2r$ basis vectors, $e_{n+2-2r}, \dots, e_{n+1}$. Thus L/Q is an ordinary Grassmannian $\text{Gr}(2r, n)$ of even-dimensional subspaces. In this way, the weights of $\mathfrak{gl}_{n+1} \cdot v$ are the last $2r$ columns of the staircase and the weights of \mathfrak{z} are the first $n-2r$ columns. We show this for $n = 8$ and $r = 2$.



Elements $\lambda \in W^Q$ act on $\Phi(\mathfrak{g}/\mathfrak{p})$ by permuting the indices of the weights $e_i + e_j$. Since

$$(e_i + e_j, e_k + e_l) = |\{i, j\} \cap \{k, l\}|,$$

we obtain the weights of $\lambda\Phi(\mathfrak{z})$ as follows. The diagonal positions in row and column i for $i = 1, \dots, n+1$ lie outside the staircase. Then λ selects $2r$ of these positions, and as before, we cross out the rows and columns of these $2r$ positions. This is displayed in Figure 5(a) for $n = 8$, $r = 2$, and $\lambda = 3569$. Then $|\pi|_\lambda$ counts the boxes in $\text{Inv}^c(\pi)$ which are not

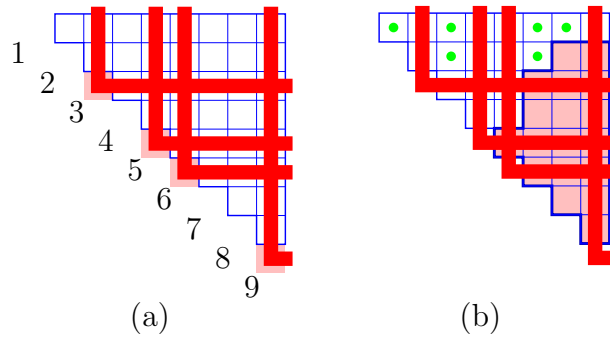


FIGURE 5. $|\pi|_\lambda = 6$ for $\pi = 8532$ and $\lambda = 3569$.

crossed out. We display this in Figure 5 for $\pi = 8532$ with the same numbers n , r , and λ as before. For this case, $|\pi|_\lambda = 6$. We associate a minimal coset representative $\lambda \in W^Q$ for $\text{Gr}(2r, n+1)$ to a selection of boxes on the diagonal in the same way as for columns in Remark 23.

We note that the inequalities for $OG(n+1)$ are quite different than the inequalities of Section 4.4 for the Lagrangian Grassmannian $LG(n)$, despite their having the same sets of solutions.

4.6. Type E_6 , G/P is the Cayley plane \mathbb{OP}^2 . This is in many ways similar to the even-dimensional quadric. Here, the parabolic subgroup P is obtained by omitting the rightmost node of the Dynkin diagram of E_6 , as shown in Table 1. Its Levi subgroup L has semisimple part Spin_{10} (type D_5), and the flag variety G/P is the even-dimensional Cayley plane \mathbb{OP}^2 . The lattice $\Phi(\mathfrak{g}/\mathfrak{p})$ is the poset \mathcal{E}_5 of Figure 6. Thus W^P is the set of (lower) order ideals in \mathcal{E}_5 , where $\pi \in W^P$ corresponds to the order ideal $\text{Inv}(\pi)$.

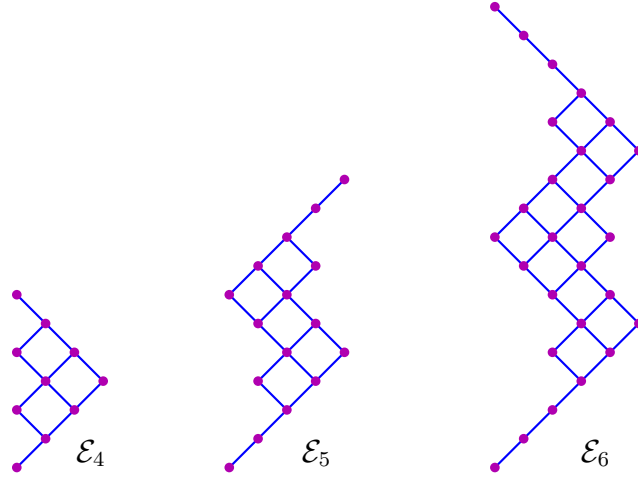


FIGURE 6. Lattices for the exceptional cominuscule flag varieties.

The tangent space $\mathfrak{g}/\mathfrak{p}$ is the 16-dimensional spinor representation of L . As in Section 4.2, $M(P)$ consists of a single parabolic subgroup Q , where $L/Q = OG(5)$. The H -fixed points on L/Q are the images of the weight spaces of $\mathfrak{g}/\mathfrak{p}$, and thus W^Q is canonically identified with \mathcal{E}_5 .

If α is the simple root defining P , then $\Phi(\mathfrak{z})$ is the orthogonal complement α^\perp to α in $\Phi(\mathfrak{g}/\mathfrak{p})$, which consists of 5 roots. Moreover for $\lambda \in W^Q$, we have $\lambda\Phi(\mathfrak{z}) = \lambda^\perp$ is the orthogonal complement to λ in $\Phi(\mathfrak{g}/\mathfrak{p})$. Consequently, viewing π as an order ideal in \mathcal{E}_5 , and λ as an element of \mathcal{E}_5 , we have the following formula:

$$|\pi|_\lambda = |\{\beta \in \Phi(\mathfrak{g}/\mathfrak{p}) \mid \beta \notin \pi, \beta \perp \lambda\}|,$$

and the inequalities (8) are

$$\sum_{i=1}^s |\pi_i|_{\lambda_i} \leq 5.$$

Note that the weight lattice $\Phi(\mathfrak{l}/\mathfrak{q})$ is isomorphic to \mathcal{E}_4 . There is a unique isomorphism from \mathcal{E}_5 to $J(\mathcal{E}_4)$. Thus we can view each $\lambda \in W^Q$ as an order ideal in \mathcal{E}_4 , which is a strict partition inside a staircase diagram. This allows us to continue the recursion with $OG(5)$, as discussed in Section 4.5.

4.7. Type E_7 , G/P is $G_\omega(\mathbb{O}^3, \mathbb{O}^6)$. The parabolic subgroup P is obtained by omitting the rightmost node of the Dynkin diagram of E_7 , as shown in Table 1, Its Levi subgroup L has type E_6 and the flag variety $G/P = G_\omega(\mathbb{O}^3, \mathbb{O}^6)$. The lattice $\Phi(\mathfrak{g}/\mathfrak{p})$ is the poset \mathcal{E}_6 , so that $\pi \in W^P$ corresponds to an order ideal in \mathcal{E}_6 , via its inversion set.

The tangent space $\mathfrak{g}/\mathfrak{p}$ is the 27-dimensional minuscule representation of E_6 . This has two interesting orbits. The smallest is the orbit through $v = v_\alpha \in \mathfrak{g}/\mathfrak{p}$, where α is the simple root defining P . It is 17-dimensional, and gives rise to the parabolic subgroup $Q \subset L$ which obtained by omitting the rightmost node of the E_6 Dynkin diagram. The second orbit is 26-dimensional, and is the orbit through $v = v_\alpha + v_{\alpha_2}$, where $\alpha_2 \in \Phi(\mathfrak{g}/\mathfrak{p})$ is the (unique) lowest root orthogonal to α . This orbit gives rise to the parabolic subgroup $Q \subset L$ obtained by omitting the leftmost node of the E_6 Dynkin diagram. Thus in both cases L/Q is isomorphic to the Cayley plane $\mathbb{O}\mathbb{P}^2$, but these two manifestations of the Cayley plane give rise to different inequalities. (This also occurs for $LG(n)$, where we have different inequalities coming from isomorphic varieties $\text{Gr}(r, n)$ and $\text{Gr}(n-r, n)$.)

As in Section 4.6, the Schubert positions for $\mathbb{O}\mathbb{P}^2$ correspond to order ideals in \mathcal{E}_5 . Since $J(\mathcal{E}_5)$ is canonically isomorphic to \mathcal{E}_6 , we will now identify W^Q with \mathcal{E}_6 .

For the smaller orbit, $\Phi(\mathfrak{z})$ is the orthogonal complement α^\perp to α in $\Phi(\mathfrak{g}/\mathfrak{p})$, which consists of 10 roots. Thus viewing π as an order ideal in \mathcal{E}_6 , and λ as an element of \mathcal{E}_6 , we have the following formula:

$$|\pi|_\lambda = |\{\beta \in \Phi(\mathfrak{g}/\mathfrak{p}) \mid \beta \notin \pi, \beta \perp \lambda\}|.$$

and the inequalities (8) for this orbit are

$$\sum_{i=1}^s |\pi_i|_{\lambda_i} \leq 10.$$

For the larger orbit, $\Phi(\mathfrak{z})$ is the orthogonal complement to $\{\alpha, \alpha_2\}$, which consists of highest root in $\Phi(\mathfrak{g}/\mathfrak{p})$. Let $\lambda \mapsto \hat{\lambda}$ denote the unique order reversing involution on \mathcal{E}_6 . Then $\Phi(\mathfrak{z}) = \hat{\alpha}$, and in general $\lambda\Phi(\mathfrak{z})$ is the single root $\hat{\lambda}$. Thus we have

$$|\pi|_\lambda = \begin{cases} 0 & \text{if } \hat{\lambda} \in \pi \\ 1 & \text{otherwise} \end{cases}$$

and the inequalities (8) for this orbit are

$$\sum_{i=1}^s |\pi_i|_{\lambda_i} \leq 1.$$

5. COMPARISON WITH OTHER INEQUALITIES

We first discuss how the classical Horn inequalities arise from the inequalities of Theorem 2 and how to modify the proof of Theorem 4 to prove their sufficiency. Next, we show how to use Proposition 11 to derive a different set of necessary inequalities for feasibility on G/P , which we call the naive inequalities. When G/P is the classical Grassmannian, these include the Horn inequalities and were essentially derived by Fulton [9, Section 1].

Our derivation of naive inequalities generalizes Theorem 36 of Belkale and Kumar in [3]. While their subset is a proper subset of the inequalities (7) from Theorem 2, it includes none of the sufficient inequalities (8).

Finally, we explain these naive inequalities in detail for the Lagrangian Grassmannian, which shows they are quite different than the inequalities of Theorem 4, as given in Section 4.4. We conjecture that the naive inequalities are sufficient to determine feasibility for the Lagrangian Grassmannian. We have verified this conjecture for $s = 3$ and $n \leq 8$.

5.1. Horn inequalities. Schubert classes σ_μ in the cohomology of the Grassmannian $\text{Gr}(k, n)$ are traditionally indexed by partitions μ , which are weakly decreasing sequences of non-negative integers

$$\mu : n-k \geq \mu^1 \geq \mu^2 \geq \cdots \geq \mu^k \geq 0.$$

Write $|\mu|$ for the sum $\mu^1 + \cdots + \mu^k$. The partition μ associated to a Schubert position π is essentially its coinversion set $\text{Inv}^c(\pi)$. Specifically, μ^i is the number of positive roots of the form $e_j - e_i$ which are coinversions. With the conventions of Section 4.1, the Ferrers diagram of μ is the reflection of $\text{Inv}^c(\pi)$ across a vertical line.

Let μ^t denote the conjugate partition to μ , whose Ferrers diagram is obtained by transposing the Ferrers diagram of μ . Note that if μ indexes a Schubert class for $\text{Gr}(k, n)$, then μ^t indexes a Schubert class for $\text{Gr}(n-k, n)$.

Given Schubert positions μ_1, \dots, μ_m and ν for $\text{Gr}(k, n)$, we say that σ_ν *occurs in* $\prod_{i=1}^m \sigma_{\mu_i}$ if, when we expand the product in the basis of Schubert classes, σ_ν occurs with a non-zero coefficient. Necessarily, we must have the codimension condition

$$|\nu| = |\mu_1| + |\mu_2| + \cdots + |\mu_m|.$$

If μ is a partition indexing a Schubert position for $\text{Gr}(k, n)$, and $\kappa : k-r \geq \kappa^1 \geq \cdots \geq \kappa^r \geq 0$ is a partition for $\text{Gr}(r, k)$, let

$$\kappa[a] := a + \kappa^{r+1-a} \quad \text{and} \quad |\mu|^\kappa := \sum_{a=1}^r \mu^{\kappa[a]}.$$

We recall the Horn recursion for $\text{Gr}(k, n)$, following Fulton [8, Theorem 17(1)].

Proposition 24. *Let μ_1, \dots, μ_m and ν be Schubert positions for $\text{Gr}(k, n)$ with $|\nu| = |\mu_1| + \cdots + |\mu_m|$. The following are equivalent.*

- (i) σ_ν occurs in $\prod_{i=1}^m \sigma_{\mu_i}$.
- (ii) The inequality

$$(26) \quad \sum_{i=1}^m |\mu_i|^{\kappa_i} \geq |\nu|^\theta$$

holds for all Schubert positions $\kappa_1, \dots, \kappa_m$ and θ for $\text{Gr}(r, k)$ such that σ_θ occurs in $\prod_{i=1}^m \sigma_{\kappa_i}$, and all $1 \leq r < k$.

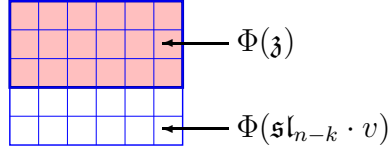
The proof of Theorem 4 can be modified to prove Proposition 24.

As we saw in Section 4.1, the semisimple part of the Levi subgroup is a product $L^{ss} = L_0 \times L_1$. Rather than study the tangent space $\mathfrak{l} \cdot v$ to the L -orbit through v , we instead study the tangent space $\mathfrak{l}_1 \cdot v$ to the L_1 -orbit through v . Then Lemma 13 is true under this substitution for the following reason. Let ϕ_1 denote the new quotient map $\phi_1: (\mathfrak{g}/\mathfrak{p}) \rightarrow (\mathfrak{g}/\mathfrak{p})/(\mathfrak{l}_1 \cdot v)$. We know from Lemma 13 (as originally stated) that the intersection $\bigcap_{i=1}^s l_i T_{\pi_i}$ is transverse if and only if $\bigcap_{i=1}^s \phi(l_i T_{\pi_i})$ is transverse. But since ϕ factors through ϕ_1 , by Proposition 1(i) these are transverse if and only if $\bigcap_{i=1}^s \phi_1(l_i T_{\pi_i})$ is transverse. The rest of the proof proceeds very much as written (although most of the Appendix is unnecessary since this is type A). We deduce that by using only one factor of L^{ss} , one obtains a set of necessary and sufficient inequalities for feasibility on Grassmannians, different from those of Theorem 4.

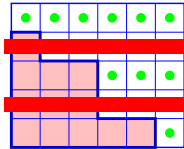
These inequalities turn out to be the classical Horn inequalities. To see this, we adopt some of the notation of Section 4.1, identifying $\mathfrak{g}/\mathfrak{p}$ with $\text{Hom}(\mathbb{C}^k, \mathbb{C}^{n-k})$ and L^{ss} with $SL_k(\mathbb{C}) \times SL_{n-k}(\mathbb{C})$, where $L_1 = SL_{n-k}(\mathbb{C})$. If v has the form (23), then

$$\Phi(\mathfrak{sl}_{n-k} \cdot v) = \Phi(\mathfrak{l}_1 \cdot v) = \{e_j - e_i \mid k - r < i \leq k < j \leq n\}.$$

Thus $\Phi(\mathfrak{z})$ is the upper $(k-r) \times (n-k)$ rectangle in $\Phi(\mathfrak{g}/\mathfrak{p})$, so that $\dim \mathfrak{z} = (k-r)(n-k)$. We display this when $n = 11$, $k = 5$, and $r = 2$.



The subgroup Q which is the stabilizer of $\mathfrak{sl}_{n-k} \cdot v$ is obtained by further omitting the node at position $k-r$ in the Dynkin diagram for L . Thus L/Q is isomorphic to $\text{Gr}(k-r, k)$. Elements $\lambda \in W^Q$ act on $\Phi(\mathfrak{g}/\mathfrak{p})$ by shuffling the r rows which do not meet $\Phi(\mathfrak{z})$ with those that do. As before, $|\pi|_\lambda$ is the number of boxes in $\text{Inv}^c(\pi)$ which remain after crossing out the images of the rows not in \mathfrak{z} . For example, when $n = 11$, $k = 5$, $\pi = 1367\ 10\ 24589\ 11$, $r = 2$, and we select rows 2 and 4 from the top, we see that $|\pi|_\lambda = 10$.



The preceding discussion shows that we have the following recursion for top-degree Schubert positions (the analog of Corollary 5).

Proposition 25 (Horn recursion). *Let π_1, \dots, π_s be a top-degree Schubert position for $\text{Gr}(k, n)$. Then π_1, \dots, π_s is feasible if and only if for every $1 \leq r < k$ and every feasible*

top-degree Schubert position $\lambda_1, \dots, \lambda_s$ for $\text{Gr}(k-r, k)$, we have

$$(27) \quad \sum_{i=1}^s |\pi_i|_{\lambda_i} \leq (k-r)(n-k)$$

Finally, we show that the two recursions in Propositions 24 and 25 are identical.

Let μ be the partition associated to π and κ^t be the partition associated to λ ; thus κ is a partition for $\text{Gr}(r, k)$. If we compare the definition of $|\pi|_\lambda$ with Remark 23, which explains how to associate an inversion diagram to the rows selected by $\lambda \in W^Q$, we see that

$$(28) \quad |\pi|_\lambda = |\mu| - |\mu|^\kappa.$$

Let ν be a partition for a Schubert position for $\text{Gr}(k, n)$. The dual partition $\widehat{\nu}$ defined by

$$\widehat{\nu}^a = n - k - \nu^{k+1-a},$$

has the property that $|\nu| + |\widehat{\nu}| = k(n-k)$ and

$$\int_{\text{Gr}(k,n)} \sigma_\mu \sigma_{\widehat{\nu}} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Thus σ_ν appears in $\prod_{i=1}^m \sigma_{\mu_i}$ if and only if $\mu^1, \dots, \mu^m, \widehat{\nu}$ is a feasible top-degree Schubert position.

The reader can easily verify that $|\nu|^\theta = r(n-k) - |\widehat{\nu}|^{\widehat{\theta}}$. Thus (26) becomes

$$(29) \quad \sum_{i=1}^m |\mu_i|^{\kappa_i} + |\widehat{\nu}|^{\widehat{\theta}} \geq r(n-k).$$

Since $\mu_1, \dots, \mu_m, \widehat{\nu}$ is a top-degree Schubert position for $\text{Gr}(k, n)$,

$$|\mu_1| + |\mu_2| + \dots + |\mu_m| + |\widehat{\nu}| = k(n-k).$$

We subtract (29) from this, setting $s := m+1$, $\mu^s := \widehat{\nu}$, and $\kappa^s := \widehat{\theta}$, to obtain

$$\sum_{i=1}^s (|\mu_i| - |\mu_i|^{\kappa_i}) \leq (k-r)(n-k).$$

If the partition μ_i corresponds to the representative $\pi_i \in W^P$ and the partition κ_i^t to the representative $\lambda_i \in W^Q$, then, by (28), this is just the condition (27).

5.2. Naive inequalities. Recall the situation of Proposition 11. We have parabolic subgroups $R \subset P \subset G$ and a correspondence between Schubert positions λ for $P/R (= L/Q)$, π for G/P , and $\lambda\pi$ for G/R .

Suppose that P' is another parabolic subgroup of G which contains R . The image of the Schubert variety $X_{\lambda\pi}$ of G/R under the projection to G/P' is a (translate of a) Schubert variety $X_{\pi'}$ of G/P' . Write $\|\pi\|_\lambda$ for the codimension of $X_{\pi'}$ in G/P' . We intentionally suppress the dependence of π' on λ and of $\|\pi\|_\lambda$ on P' . We use Proposition 11, which relates feasibility for Schubert problems on different flag varieties, to obtain necessary inequalities which hold for feasible Schubert problems on G/P .

Theorem 26. *Suppose that π_1, \dots, π_s is a feasible Schubert position for G/P . Given parabolic subgroups $R \subset P'$ of G with $R \subset P$ and any feasible Schubert position $\lambda_1, \dots, \lambda_s$ for P/R , the Schubert position π'_1, \dots, π'_s is feasible for G/P' . In particular, any necessary inequalities for feasibility on G/P' give inequalities on the original Schubert position π_1, \dots, π_s for G/P . For example, the basic codimension inequality for π'_1, \dots, π'_s gives*

$$(30) \quad \sum_i \|\pi_i\|_{\lambda_i} \leq \dim G/P'.$$

Proof. By Proposition 11(ii), $\lambda_1\pi_1, \dots, \lambda_s\pi_s$ is a feasible Schubert position for G/R , and so by Proposition 11(i), π'_1, \dots, π'_s is a feasible Schubert position for G/P' . The rest is immediate. \square

Remark 27. Belkale and Kumar [3, Theorem 36] use similar ideas to also derive (30). When $P' \cap P = R$, they express these inequalities in a form similar to (21), in terms of counting roots [3, inequality (58)]. In fact, these are the inequalities of Theorem 2, when $\mathfrak{s} = \mathfrak{n}_{P'} \cap \mathfrak{n}_P$. Since this is almost never the center of \mathfrak{n}_R , none of the inequalities of Belkale and Kumar have the form (8).

We note that the inequalities (30) are always a subset of the necessary inequalities of Theorem 2. The verification of this assertion is left as an exercise.

Remark 28. Theorem 26 gives a method to generate many necessary inequalities for feasibility on different flag varieties G/P . For example, in type A we can take $R = P$ and let P' be any maximal parabolic subgroup containing P . Then G/P' is a classical Grassmannian and Theorem 26 shows how to pull back the Horn inequalities for G/P' to obtain inequalities for G/P .

If G has type A , B , C , or D and P is a maximal parabolic subgroup, then we can select $R \subset P$ so that P/R is a classical Grassmannian. If P' is a different parabolic subgroup of G which contains R , then the codimension inequalities (30) give necessary inequalities for feasibility on G/P which are indexed by feasible Schubert problems on a classical Grassmannian P/R .

We invite the reader to check that in type A , this last procedure is yet another method for deriving the necessity of the Horn inequalities. In fact, Fulton essentially did just that in [9, Section 1].

We also invite the reader to use Theorem 26 to generate even more necessary inequalities for feasibility on flag varieties G/P . We believe that it is an interesting and worthwhile project to investigate these naive Horn-type inequalities on other flag varieties. For example, for which flag varieties is (a natural subset of) the set of all such naive inequalities sufficient to determine feasibility? In the next section, we examine a subset of these in detail for the Lagrangian Grassmannian, showing that they are in general different than the necessary and sufficient inequalities derived in Section 4.4.

5.3. Naive inequalities for the Lagrangian Grassmannian. We express codimension inequalities (30) of Theorem 26 for the Lagrangian Grassmannian in a form similar to the inequalities of Corollary 5.

Theorem 29. *Let π_1, \dots, π_s be a feasible top-degree Schubert position for the Lagrangian Grassmannian $LG(n)$. Then, for any feasible Schubert positions $\lambda_1, \dots, \lambda_s$ for $\text{Gr}(r, n)$,*

we have

$$(31) \quad \sum_{i=1}^s |\pi_i|_{\widehat{\lambda}_i^t} \geq \binom{r+1}{2}.$$

Here, $\widehat{\lambda}^t$ is the conjugate of the dual Schubert position to λ , as in Section 5.1.

Note that $\lambda_1, \dots, \lambda_s$ is feasible for $\text{Gr}(r, n)$ if and only if $\lambda_1^t, \dots, \lambda_s^t$ is feasible for $\text{Gr}(n-r, n)$. Thus the inequalities (31) may be rewritten

$$\sum_{i=1}^s |\pi_i|_{\widehat{\lambda}_i} \geq \binom{n-r+1}{2}.$$

These bear a striking similarity to the inequalities of Corollary 5 for the Lagrangian Grassmannian, which by the discussion in Section 4.4, have the form

$$\sum_{i=1}^s |\pi_i|_{\lambda_i} \leq \binom{n-r+1}{2},$$

and are indexed by the same set as the necessary inequalities of Theorem 29. In fact these inequalities are quite different. Not only does the inequality go in the opposite direction, but the terms $|\pi_i|_{\widehat{\lambda}_i}$ and $|\pi_i|_{\lambda_i}$ are unrelated quantities.

Proof. Let $G = Sp_{2n}(\mathbb{C})$ and P_k be the maximal parabolic subgroup corresponding to the k th simple root from the right end of the Dynkin diagram of C_n as shown in Table 1. Then G/P_k is a space of isotropic k -dimensional linear subspaces of a \mathbb{C}^{2n} which is equipped with a non-degenerate alternating bilinear form and $\dim G/P_k = 2k(n-k) + \binom{k+1}{2}$.

We consider the codimension inequalities (30) of Theorem 26 for $G/P_n = LG(n)$, the Lagrangian Grassmannian, $P' = P_{n-r}$, and $R = P_n \cap P_{n-r}$. Let π be a Schubert position for $LG(n)$ and λ a Schubert position for $P_n/R = \text{Gr}(r, n)$. (Note: it is consistent with the conventions established in Section 4.4, to call this $\text{Gr}(r, n)$, rather than $\text{Gr}(n-r, n)$.)

We will show

$$(32) \quad \|\pi\|_{\lambda} = |\pi| + |\lambda| - |\pi|_{\widehat{\lambda}^t}.$$

Then (31) will follow, for

$$\begin{aligned} \sum_{i=1}^s |\pi_i|_{\widehat{\lambda}_i^t} &= \sum_{i=1}^s |\pi_i| + \sum_{i=1}^s |\lambda_i| - \sum_{i=1}^s \|\pi_i\|_{\lambda_i} \\ &\geq \binom{n+1}{2} + r(n-r) - 2r(n-r) - \binom{n-r+1}{2} = \binom{r+1}{2}. \end{aligned}$$

Indeed, $\sum_i |\pi_i| = \binom{n+1}{2}$ and $\sum_i |\lambda_i| = r(n-r)$, as these are top-degree Schubert positions, and the inequality comes from the negative of inequality (30).

We deduce (32) using a uniform combinatorial model for Schubert positions in these flag varieties, which may be found in [11]. Schubert positions w for G/P_k are represented by increasing sequences of integers

$$w : 1 \leq w^1 < w^2 < \dots < w^k \leq 2n,$$

where we do not have $w^i + w^j = 2n + 1$ for any i, j . (The corresponding Schubert variety consists of those isotropic k -planes V where $\dim V \cap F^{2n+1-w^i} \geq i$, where F^1, F^2, \dots is a fixed isotropic flag with $i = \dim F^i$.) Then

$$(33) \quad |w| := \left(\sum_{j=1}^k w^j - j \right) - |\{a < b \mid w^a + w^b > 2n + 1\}|.$$

Let $k = n$ and recall the conventions of Section 4.4 for drawing coinversion sets for $\pi \in W^{P_n}$ as strict partitions in the staircase shape with diagonal boxes (and hence rows and columns) labeled $1, \dots, n$. Let w be the increasing sequence of integers corresponding to $\pi \in W^{P_n}$. The correspondence is such that π has a coinversion in position $(n+1-a, n+1-b)$ if and only $w^a + w^b > 2n + 1$. The term $w^j - j$ in (33) counts the number of coinversions in the hook through row and column $n+1-j$, while $|\{a < b \mid w^a + w^b > 2n + 1\}|$ is the total number of off-diagonal coinversions, which are counted twice in the sum.

Let κ^t be the partition corresponding to a Schubert position λ for $\text{Gr}(r, n)$; thus κ indexes a Schubert position for $\text{Gr}(n-r, n)$. Recall that $\kappa[a] := a + \kappa^{n-r+1-a}$. If we lift a Schubert position w for G/P_n to G/R using κ^t and then project to G/P_{n-r} , we obtain the Schubert position

$$w' := w^{\kappa[1]} < w^{\kappa[2]} < \dots < w^{\kappa[n-r]},$$

and so

$$\|\pi\|_\lambda = |w'| = \left(\sum_{j=1}^{n-r} w^{\kappa[j]} - j \right) - |\{a < b \mid w^{\kappa[a]} + w^{\kappa[b]} > 2n + 1\}|.$$

Consider $\|\pi\|_\lambda - |\lambda| = \|\pi\|_\lambda - \sum_{j=1}^s \kappa_{n-r+1-j}$, which is

$$\left(\sum_{j=1}^{n-r} w^{\kappa[j]} - \kappa[j] \right) - |\{a < b \mid w^{\kappa[a]} + w^{\kappa[b]} > 2n + 1\}|.$$

From the discussion interpreting the terms of (33) for $LG(n)$, it follows that this sum is the number of coinversions of π which lie in the hooks through rows and columns indexed $n+1-\kappa[j]$, for $j = 1, \dots, n-r$. The subtracted term $|\{a < b \mid w^{\kappa[a]} + w^{\kappa[b]} > 2n + 1\}|$ is the number of such coinversions counted twice by the sum. From the definition of $\hat{\kappa}$ these are the rows and columns indexed by $\hat{\kappa}[j]$, for $j = 1, \dots, n-r$. But this is just $|\pi| - |\pi|_{\hat{\kappa}}$, which proves (32). \square

APPENDIX A. ROOT SYSTEM MISCELLANY

Our situation and notation will be as in the proof of Theorem 4. To recap, suppose that G is a reductive algebraic group for which G^{ss} is simple, and let $P \subset G$ be a parabolic subgroup so that G/P is cominuscule. We will freely use the characterizations (i)–(iv) of cominuscule flag varieties from Section 1.4. Fix a maximal torus H of P and let L be the Levi subgroup of P which contains H . Let $v \in \mathfrak{g}/\mathfrak{p}$, and assume v is neither 0, nor in the dense orbit of L on $\mathfrak{g}/\mathfrak{p}$. Let $Q \subset L$ be the stabilizer of $\mathfrak{l} \cdot v$, and define \mathfrak{z} to be the quotient of the tangent space $\mathfrak{g}/\mathfrak{p}$ by $\mathfrak{l} \cdot v$.

We establish some essential facts about the root-space decomposition of the Lie algebras $\mathfrak{g}, \mathfrak{p}, \mathfrak{l}, \mathfrak{q}$, etc., as well as the subquotients $\mathfrak{l} \cdot v$ and \mathfrak{z} . These results are needed in the proof of Theorem 4. We begin with some general statements.

Throughout, roots will mean the roots of \mathfrak{g} . Let Φ be the set of roots of \mathfrak{g} , which are the non-zero weights of \mathfrak{g} under the action of the maximal torus H . Once and for all, choose a non-zero vector $v_\beta \in \mathfrak{g}_\beta$ in each weight space of \mathfrak{g} . If \mathfrak{s} is a subquotient H -module of \mathfrak{g} , then we write $\Phi(\mathfrak{s}) \subset \Phi$ for the non-zero weights of \mathfrak{s} . If $\beta \in \Phi(\mathfrak{s})$, then we also write $v_\beta (\in \mathfrak{s})$ for the image of $v_\beta \in \mathfrak{g}$ in \mathfrak{s} .

If $\Delta \subset \Phi$ is a system of simple roots, then we may express any root $\beta \in \Phi$ uniquely as an integral linear combination of the simple roots in Δ . Let $m_\delta(\beta)$ be the coefficient of $\delta \in \Delta$ in this expression for β . Write $m_\delta(\mathfrak{g})$ for $\max_{\beta \in \Phi(\mathfrak{g})} (m_\delta(\beta))$, which is the coefficient of δ in the highest root of \mathfrak{g} .

A root β is *positive* (respectively *negative*) if any coefficient $m_\alpha(\beta)$ is positive (respectively negative). Since a root cannot be both positive and negative, we have the decomposition $\Phi = \Phi^+ \sqcup \Phi^-$ of Φ into positive and negative roots. We say that a root β is higher than β' if $\beta - \beta'$ is a positive root. This definition depends upon the choice Δ of simple roots. We say that Δ is *compatible* with P if $\Phi^- \subset \Phi(\mathfrak{p})$. If $\alpha \in \Delta$ is the root defining P , $\{\alpha\} = \Delta - \Phi(\mathfrak{p})$, then the weights of \mathfrak{l} are exactly those $\beta \in \Phi$ such that $m_\alpha(\beta) = 0$.

Recall that the standard pairing on the root space \mathfrak{h}^* is

$$\langle \beta, \alpha \rangle := 2 \frac{(\beta, \alpha)}{(\alpha, \alpha)}.$$

Here, (\cdot, \cdot) is any W -invariant Euclidean inner product on \mathfrak{h}^* . By a long root, we mean any root α for which (α, α) is maximized. If G is simply laced, then every root is long.

We recall the following basic facts about root systems. Numbers 1 and 2 are found, for example, in Section 9.4 of [13].

1. If α is a long root then $\langle \beta, \alpha \rangle \in \{-2, -1, 0, 1, 2\}$ and $\langle \beta, \alpha \rangle = \pm 2$ only if $\beta = \pm \alpha$.
2. If $\beta, \alpha \in \Phi$ with $\pm \langle \beta, \alpha \rangle < 0$, then $\beta \pm \alpha$ is a root. If α is a long root and $\beta + \alpha$ is a root, then $\langle \beta, \alpha \rangle = -1$.
3. If a subgroup K of G contains the maximal torus H and \mathfrak{s} is a K -subrepresentation of \mathfrak{g} , then for every $\gamma \in \Phi(\mathfrak{s})$ and $\beta \in \Phi(\mathfrak{k})$ with $\beta + \gamma \in \Phi$, we have $\beta + \gamma \in \Phi(\mathfrak{s})$. (Here, \mathfrak{k} is the Lie algebra of K .)

Proof. If $\beta + \gamma$ is a root then v_β acts non-trivially on v_γ and the result lies in $\mathfrak{g}_{\beta+\gamma}$, and so $\mathfrak{g}_{\beta+\gamma} \subset \mathfrak{s}$. \square

Given a system $\Delta \subset \Phi$ of simple roots of \mathfrak{g} , a sequence

$$\gamma_1 \rightarrow \gamma_2 \rightarrow \cdots \rightarrow \gamma_e$$

of roots of \mathfrak{g} is an *increasing chain* if, for all k , $\gamma_{k+1} = \gamma_k + \delta_k$ where $\delta_k \in \Delta$. That is, if at each step we raise by a simple root. If $\gamma_1 \in \Delta$, then for $\delta \in \Delta$, the coefficient $m_\delta(\gamma_e)$ is the number of times δ was used in the chain (including the first step $0 \rightarrow \gamma_1$).

Lemma A.1. *Let K be any algebraic subgroup of G containing H .*

- (i) *If G/K is a cominuscule flag variety, then for any $\beta_1, \beta_2 \in \Phi(\mathfrak{g}/\mathfrak{k})$, $\langle \beta_1, \beta_2 \rangle \geq 0$.*
- (ii) *If G/K is not a cominuscule flag variety, then there exist $\beta_1, \beta_2 \in \Phi(\mathfrak{g}/\mathfrak{k})$ such that $\beta_1 + \beta_2 \in \Phi \cup \{0\}$.*

Note that (i) implies the converse of (ii) and if G is simply laced, then (ii) implies the converse of (i).

Proof. Suppose that the homogeneous space G/K is a cominuscule flag variety. Then in particular K is a maximal parabolic subgroup. Choose a system Δ of simple roots compatible with K , and let $\alpha \in \Delta$ be the simple root defining K . Since G/K is cominuscule, $m_\alpha(\beta) = 1$ for every $\beta \in \Phi(\mathfrak{g}/\mathfrak{k})$. Indeed, every root in $\Phi(\mathfrak{g}/\mathfrak{k})$ lies in an increasing chain of roots that starts with α and ends with the highest root. For (i), if $\langle \beta_1, \beta_2 \rangle < 0$ then $\beta_1 + \beta_2 \in \Phi(\mathfrak{g}/\mathfrak{k})$ and so $m_\alpha(\beta_1 + \beta_2) = m_\alpha(\beta_1) + m_\alpha(\beta_2) = 2$, which is a contradiction.

Suppose that G/K is not cominuscule. If K is not a parabolic subgroup, then there exists a root γ of \mathfrak{g} with neither γ nor $-\gamma$ a root of \mathfrak{k} . Thus we can take $\beta_1 = -\beta_2 = \gamma$. Otherwise, choose a positive system of roots compatible with K , and let γ_1 be a simple root defining K . Take an increasing chain of roots connecting γ_1 to the highest root,

$$\gamma_1 \rightarrow \gamma_2 \rightarrow \cdots \rightarrow \gamma_{\text{top}}.$$

Observe that each $\gamma_k \in \Phi(\mathfrak{g}/\mathfrak{k})$. Since G/K is not cominuscule, either there is another simple root in $\Phi(\mathfrak{g}/\mathfrak{k})$ or else $m_{\gamma_1}(\gamma_{\text{top}}) \geq 2$. Thus at some point γ_k in this chain, we will raise by a simple root $\delta \in \Phi(\mathfrak{g}/\mathfrak{k})$. Thus $\gamma_{k+1} = \gamma_k + \delta$, so we can take $\beta_1 = \gamma_k$ and $\beta_2 = \delta$. \square

An *orthogonal sequence of long roots* in $\Phi(\mathfrak{g}/\mathfrak{p})$ is a sequence $\alpha_1, \dots, \alpha_r \in \Phi(\mathfrak{g}/\mathfrak{p})$, where α_i are long roots, and $\langle \alpha_i, \alpha_j \rangle = 0$ for $i \neq j$. Such a sequence is maximal if every long root $\beta \in \Phi(\mathfrak{g}/\mathfrak{p})$ is non-orthogonal to some α_i . Orthogonal sequences of long roots play a key role in the structure of $\mathfrak{g}/\mathfrak{p}$.

If G/P is cominuscule then every non-zero vector $v \in \mathfrak{g}/\mathfrak{p}$ lies in the L -orbit of a sum $v_{\alpha_1} + \cdots + v_{\alpha_r}$, where $\alpha_1, \dots, \alpha_r \in \Phi(\mathfrak{g}/\mathfrak{p})$ is an orthogonal sequence of long roots [22]. Our assumption that v does not lie in the dense orbit is equivalent to assuming that $\alpha_1, \dots, \alpha_r$ is not maximal. The construction of Q , \mathfrak{z} , etc. is L -equivariant with respect to the choice of v , and thus we encounter no loss of generality in assuming v takes this normal form. We therefore write

$$v = v_{\alpha_1} + \cdots + v_{\alpha_r},$$

with $\langle \alpha_i, \alpha_j \rangle = 0$ for $i \neq j$. However, note that in the following lemmas, whenever α_i does not appear explicitly in the statement, the result is valid for all non-zero $v \in \mathfrak{g}/\mathfrak{p}$ which are not in the dense orbit of L .

Lemma A.2. *If $\gamma \in \Phi(\mathfrak{l})$, then there is at most one index i such that $\langle \gamma, \alpha_i \rangle = -1$.*

Proof. Suppose that $\langle \gamma, \alpha_i \rangle = -1$. Then $\gamma + \alpha_i \in \Phi$, and since L preserves $\mathfrak{g}/\mathfrak{p}$, we have $\gamma + \alpha_i \in \Phi(\mathfrak{g}/\mathfrak{p})$. If $j \neq i$, then Lemma A.1 (ii) and $\langle \alpha_i, \alpha_j \rangle = 0$ imply that

$$0 < \langle \gamma + \alpha_i, \alpha_j \rangle = \langle \gamma, \alpha_j \rangle. \quad \square$$

Lemma A.3. *Let $\alpha \in \Phi(\mathfrak{g}/\mathfrak{p})$ be a long root. Then $\mathfrak{l} \cdot v_\alpha$ is H -invariant and $\Phi(\mathfrak{l} \cdot v_\alpha) = \{\beta \in \Phi(\mathfrak{g}/\mathfrak{p}) \mid \langle \beta, \alpha \rangle \geq 1\}$.*

Proof. As G/P is cominuscule, all long roots of $\mathfrak{g}/\mathfrak{p}$ are conjugate (in fact by W_L [22]), so we may assume that $\alpha \in \Delta$ defines P . Since \mathfrak{g}_α and \mathfrak{l} are H -modules, so is $\mathfrak{l} \cdot v_\alpha = \mathfrak{l} \cdot \mathfrak{g}_\alpha$. Note that

$$\Phi(\mathfrak{l} \cdot v_\alpha) = \{\beta \in \Phi(\mathfrak{g}/\mathfrak{p}) \mid \beta - \alpha \in \Phi(\mathfrak{l}) \cup \{0\}\}.$$

Let $\beta \in \Phi(\mathfrak{g}/\mathfrak{p})$ and suppose that $\langle \beta, \alpha \rangle \geq 1$. Recall that $m_\alpha(\beta) = 1$. If $\langle \beta, \alpha \rangle = 2$, then $\beta = \alpha$. Otherwise $\langle \beta, \alpha \rangle = 1$ and so $\beta - \alpha$ is a root. Then $m_\alpha(\beta - \alpha) = m_\alpha(\beta) - 1 = 0$, and thus $\beta - \alpha \in \Phi(\mathfrak{l})$.

We show the other inclusion. If $\beta - \alpha \in \Phi(\mathfrak{l}) \cup \{0\}$ then

$$\langle \beta, \alpha \rangle = \langle \beta - \alpha, \alpha \rangle + \langle \alpha, \alpha \rangle \geq -1 + 2 = 1. \quad \square$$

Recall that $\mathfrak{z} := (\mathfrak{g}/\mathfrak{p})/(\mathfrak{l} \cdot v)$.

Lemma A.4. $\mathfrak{l} \cdot v$ is H -invariant and we have

$$\begin{aligned} \Phi(\mathfrak{l} \cdot v) &= \{ \beta \in \Phi(\mathfrak{g}/\mathfrak{p}) \mid \langle \beta, \alpha_i \rangle \geq 1, \text{ for some } i \}, \text{ and} \\ \Phi(\mathfrak{z}) &= \{ \beta \in \Phi(\mathfrak{g}/\mathfrak{p}) \mid \langle \beta, \alpha_i \rangle = 0, \text{ for all } i \}. \end{aligned}$$

This holds even when v lies in the dense orbit of L .

Proof. We claim that

$$\mathfrak{l} \cdot v = \mathfrak{l} \cdot v_{\alpha_1} + \mathfrak{l} \cdot v_{\alpha_2} + \cdots + \mathfrak{l} \cdot v_{\alpha_r},$$

from which the statement of the lemma follows from Lemma A.3, as each α_i is a long root. For each $i = 1, \dots, r$, let $\mathfrak{l}_i \subset \mathfrak{l}$ be the linear span of the set

$$\Gamma_i := \{ v_\gamma \mid \gamma \in \Phi(\mathfrak{l}) \text{ and } \langle \gamma, \alpha_i \rangle = -1 \}.$$

Then we have $\mathfrak{l} \cdot v_{\alpha_i} = \mathfrak{l}_i \cdot v_{\alpha_i} = \mathfrak{l}_i \cdot v$. The last equality is a consequence of Lemma A.2. Lemma A.2 also implies that the sets Γ_i are disjoint and therefore $\mathfrak{l}_1 + \mathfrak{l}_2 + \cdots + \mathfrak{l}_r$ is a direct sum. Thus we have

$$\begin{aligned} \mathfrak{l} \cdot v &= \mathfrak{l} \cdot (v_{\alpha_1} + v_{\alpha_2} + \cdots + v_{\alpha_r}) \\ &\subset \mathfrak{l} \cdot v_{\alpha_1} + \mathfrak{l} \cdot v_{\alpha_2} + \cdots + \mathfrak{l} \cdot v_{\alpha_r} \\ &= \mathfrak{l}_1 \cdot v_{\alpha_1} + \mathfrak{l}_2 \cdot v_{\alpha_2} + \cdots + \mathfrak{l}_r \cdot v_{\alpha_r} \\ &= \mathfrak{l}_1 \cdot v + \mathfrak{l}_2 \cdot v + \cdots + \mathfrak{l}_r \cdot v \\ &= (\mathfrak{l}_1 + \mathfrak{l}_2 + \cdots + \mathfrak{l}_r) \cdot v \subset \mathfrak{l} \cdot v, \end{aligned}$$

which proves the claim. \square

Lemma A.5. Let $\alpha_1, \alpha_2, \dots, \alpha_r \in \Phi(\mathfrak{g}/\mathfrak{p})$ be an orthogonal sequence of long roots.

- (i) For any $\beta \in \Phi(\mathfrak{g}/\mathfrak{p})$, there exist at most two distinct indices i such that $\langle \beta, \alpha_i \rangle \geq 1$.
- (ii) There exists $\beta \in \Phi(\mathfrak{g}/\mathfrak{p})$ such that $\langle \beta, \alpha_1 \rangle = \langle \beta, \alpha_2 \rangle = 1$, when $r \geq 2$.

Proof. Choose a system Δ of simple roots compatible with P and let $\alpha \in \Delta$ be the simple root defining P .

(i) Suppose there are three indices, say i, j, k . Then $\langle \beta, \alpha_i \rangle \geq 1$ so $\beta - \alpha_i$ is a root. Then $\langle \beta - \alpha_i, \alpha_j \rangle \geq 1$, so $\beta - \alpha_i - \alpha_j$ is a root. Similarly $\beta - \alpha_i - \alpha_j - \alpha_k$ is a root. But now $m_\alpha(\beta - \alpha_i - \alpha_j - \alpha_k) = -2$ and there is no root with this property as G/P is cominuscule.

(ii) Since W_L acts transitively on all orthogonal sequences of long roots of the same length [22], it suffices to show this for a particular pair of orthogonal long roots. Set $\alpha_1 := \alpha$, the simple root defining P , and let α_2 be the highest root of \mathfrak{g} . If an orthogonal pair of long roots exists, Lemma A.8 implies this is such a pair (and the argument is non-circular). Let $\delta \in \Delta$ be a root such that $\langle \delta, \alpha_2 \rangle = 1$ and consider the sum, β , of $\alpha + \delta$ with

all the simple roots in the Dynkin diagram of G which lie strictly between α and δ . Such a sum is always a root. We have $\langle \beta, \alpha_1 \rangle = \langle \beta, \alpha_2 \rangle = 1$. \square

Recall that $Q = \text{Stab}_L(\mathfrak{l} \cdot v)$ so that \mathfrak{q} is spanned by those v_γ which stabilize $\mathfrak{l} \cdot v$.

Lemma A.6. $\Phi(\mathfrak{q}) = \{\gamma \in \Phi(\mathfrak{l}) \mid \langle \gamma, \alpha_i \rangle = 0 \text{ for all } i \text{ or } \langle \gamma, \alpha_i \rangle \geq 1 \text{ for some } i\}$.

Observe that by Lemma A.2 we deduce,

$$\Phi(\mathfrak{l}/\mathfrak{q}) = \{\gamma \in \Phi(\mathfrak{l}) \mid \langle \gamma, \alpha_i \rangle \leq 0 \text{ for all } i \text{ and } \langle \gamma, \alpha_i \rangle = -1 \text{ for exactly one } i\}.$$

Proof. Let $\gamma \in \Phi(\mathfrak{l})$ and suppose that $\langle \gamma, \alpha_i \rangle \geq 0$, for all $i = 1, \dots, r$. If $\beta \in \Phi(\mathfrak{l} \cdot v)$, then Lemma A.4 implies that $\langle \beta, \alpha_i \rangle \geq 1$ for some i , and therefore $\langle \gamma + \beta, \alpha_i \rangle \geq 1$. Similarly, suppose that $\langle \gamma, \alpha_i \rangle \geq 1$ for some index i . If $\beta \in \Phi(\mathfrak{l} \cdot v) (\subset \Phi(\mathfrak{g}/\mathfrak{p}))$, then Lemma A.1(i) implies that $\langle \beta, \alpha_i \rangle \geq 0$, and so again we have $\langle \gamma + \beta, \alpha_i \rangle \geq 1$. Thus in either case, Lemma A.4 implies that if $\gamma + \beta$ is a root, then it lies in $\Phi(\mathfrak{l} \cdot v)$, and so $v_\gamma \in \mathfrak{q}$ as it stabilizes $\mathfrak{l} \cdot v$.

For the converse, suppose that $\langle \gamma, \alpha_i \rangle \leq 0$ for all i , and $\langle \gamma, \alpha_j \rangle = -1$ for some index j with $1 \leq j \leq r$. Suppose moreover that v_γ stabilizes $\mathfrak{l} \cdot v$. We show this leads to a contradiction.

Claim. Assume that $\beta \in \Phi(\mathfrak{g}/\mathfrak{p})$ is not equal to α_j . Then $\langle \beta, \gamma \rangle < 0$ only if there exist exactly two indices i such that $\langle \beta, \alpha_i \rangle \geq 1$.

Proof of Claim. Suppose that $\langle \beta, \gamma \rangle < 0$. Then $\gamma + \beta$ is a root, and since $\mathfrak{g}/\mathfrak{p}$ is an \mathfrak{l} -module, $\gamma + \beta \in \Phi(\mathfrak{g}/\mathfrak{p})$. By Lemma A.1(i) we have $0 \leq \langle \gamma + \beta, \alpha_j \rangle = -1 + \langle \beta, \alpha_j \rangle$. Since $\beta \neq \alpha_j$, we have $\langle \beta, \alpha_j \rangle = 1$, and so $\beta \in \Phi(\mathfrak{l} \cdot v)$, by Lemma A.4.

Since v_γ stabilizes $\mathfrak{l} \cdot v$, we must have that $\gamma + \beta \in \mathfrak{l} \cdot v$, and thus there is some index i with $\langle \gamma + \beta, \alpha_i \rangle \geq 1$. Then

$$1 \leq \langle \gamma + \beta, \alpha_i \rangle = \langle \gamma, \alpha_i \rangle + \langle \beta, \alpha_i \rangle \leq \langle \beta, \alpha_i \rangle,$$

so $1 \leq \langle \beta, \alpha_i \rangle$. Necessarily, $i \neq j$ as $\langle \gamma + \beta, \alpha_j \rangle = \langle \gamma, \alpha_j \rangle + \langle \beta, \alpha_j \rangle = 0$. \square

Now since $L \cdot v$ is not dense, there exists a long root $\alpha \in \Phi(\mathfrak{g}/\mathfrak{p})$ orthogonal to α_i for all indices i . We may assume that α is the simple root defining P .

Consider the set $h := \{\beta \in \Phi(\mathfrak{g}/\mathfrak{p}) \mid \langle \beta, \alpha \rangle > 0\}$. By Lemma A.5(i) if $\beta \in h$ then we have $\langle \beta, \alpha_i \rangle > 0$ for at most one index i (β is already positively paired with α). Also, $\alpha_j \notin h$, as $\langle \alpha_j, \alpha \rangle = 0$. Thus by the claim, we have $\langle \beta, \gamma \rangle \geq 0$ for all $\beta \in h$.

Now consider the sum, β_0 , of α and all the simple roots in the Dynkin diagram of G which lie strictly between α and the nearest simple root used in γ . Such a sum is always a root, and $\beta_0 \in \Phi(\mathfrak{g}/\mathfrak{p})$. Moreover, $\langle \beta_0, \alpha \rangle = 1$, so $\beta_0 \in h$, but $\langle \beta_0, \gamma \rangle < 0$, a contradiction. \square

In summary, the roots of \mathfrak{g} decompose into a disjoint union of

$$\Phi(\mathfrak{r}) = \Phi(\mathfrak{n}_P) \sqcup \Phi(\mathfrak{q}), \quad \Phi(\mathfrak{l}/\mathfrak{q}), \quad \Phi(\mathfrak{l} \cdot v), \quad \text{and} \quad \Phi(\mathfrak{z}).$$

The roots, γ , of these pieces are characterized by their pairings with respect to the long roots α_i and the values of $m_\alpha(\gamma)$, where α is the simple root defining P . These characterizations are given concisely in Table 2.

Lemma A.7. *The subgroup Q is a parabolic subgroup of L , and the flag variety L/Q is cominuscule.*

γ	$m_\alpha(\gamma)$	$\langle \gamma, \alpha_i \rangle$
$\Phi(\mathfrak{z})$	1	$= 0$ for all i
$\Phi(\mathfrak{l} \cdot v)$	1	≥ 0 for all i and ≥ 1 for some i
$\Phi(\mathfrak{l}/\mathfrak{q})$	0	≤ 0 for all i and $= -1$ for exactly one i
$\Phi(\mathfrak{q})$	0	$= 0$ for all i or ≥ 1 for some i
$\Phi(\mathfrak{n}_P)$	-1	≤ 0 for all i

TABLE 2. Summary of decomposition of Φ

Proof. Suppose that L/Q is not cominuscle. By Lemma A.1(ii), there must be two roots $\gamma_1, \gamma_2 \in \Phi(\mathfrak{l}/\mathfrak{q})$ with $\gamma_1 + \gamma_2 \in \Phi(\mathfrak{l}) \cup \{0\}$

By Lemma A.6, $\langle \gamma_1, \alpha_i \rangle, \langle \gamma_2, \alpha_i \rangle \leq 0$ for all i and there exist unique indices i_1 and i_2 such that

$$\langle \gamma_1, \alpha_{i_1} \rangle = -1 \text{ and } \langle \gamma_2, \alpha_{i_2} \rangle = -1.$$

If $\gamma_1 = -\gamma_2$, then $\langle \gamma_2, \alpha_{i_1} \rangle = 1$, which contradicts Lemma A.6. Otherwise, $\gamma_1 + \gamma_2$ is a root, necessarily in $\Phi(\mathfrak{l}/\mathfrak{q})$. If $i_1 = i_2$ then $\langle \gamma_1 + \gamma_2, \alpha_{i_1} \rangle = -2$, which is impossible. Otherwise we have $\langle \gamma_1 + \gamma_2, \alpha_{i_1} \rangle \leq -1$ and $\langle \gamma_1 + \gamma_2, \alpha_{i_2} \rangle \leq -1$, which is also impossible, by Lemma A.2. \square

Let R be the parabolic subgroup of P corresponding to $Q \subset L$ so that $R = \text{Stab}_P(\mathfrak{l} \cdot v)$. Since $\mathfrak{l} \cdot v$ is H -stable, $H \subset Q \subset R$. We also assume that our system Δ of simple roots is compatible with the parabolic subgroups P and R .

Lemma A.8. *The highest root of \mathfrak{g} is an element of $\Phi(\mathfrak{z})$. If $\gamma \in \Phi(\mathfrak{l})$ is a simple root defining Q , then $m_\gamma(\beta) = m_\gamma(\mathfrak{g})$ for all $\beta \in \Phi(\mathfrak{z})$, and \mathfrak{z} is an irreducible L_Q -module. The same statements hold for R in place of Q .*

Proof. Let $\beta_1 \in \Phi(\mathfrak{z}) \subset \Phi(\mathfrak{g}/\mathfrak{p})$ and consider an increasing chain of roots

$$(34) \quad \beta_1 \rightarrow \beta_2 \rightarrow \cdots$$

Let δ_k be the simple root $\delta_k := \beta_{k+1} - \beta_k$. Let α be the simple root defining P . Then $1 = m_\alpha(\beta_1) \leq m_\alpha(\beta_k) \leq m_\alpha(\mathfrak{g}) = 1$. Hence $\beta_k \in \Phi(\mathfrak{g}/\mathfrak{p})$ and $\delta_k \neq \alpha$, and we conclude that δ_k is a simple root of L .

Suppose δ_k is a simple root of L_Q . Since $\mathfrak{l} \cdot v$ is a Q -submodule of $\mathfrak{g}/\mathfrak{p}$, we have the decomposition $\mathfrak{g}/\mathfrak{p} = (\mathfrak{l} \cdot v) \oplus \mathfrak{z}$ as L_Q -modules (since L_Q is reductive). In particular \mathfrak{z} is an L_Q -module. Thus if $\beta_k \in \Phi(\mathfrak{z})$, then $\beta_{k+1} = \beta_k + \delta_k \in \Phi(\mathfrak{z})$.

Otherwise, $\delta_k = \gamma$, a simple root of L defining Q . Then $m_\gamma(\beta_{k+1}) = m_\gamma(\beta_k) + 1$. By Lemma A.6, there is some index i such that $\langle \gamma, \alpha_i \rangle = -1$, as $\gamma \in \Phi(\mathfrak{l}/\mathfrak{q})$. Then Lemma A.4 implies that $\langle \beta_{k+1}, \alpha_i \rangle = \langle \beta_k, \alpha_i \rangle - 1 = -1$, which contradicts Lemma A.1(i).

We conclude that every simple root δ_k arising from our chain (34) is a simple root of Q , and each $\beta_k \in \Phi(\mathfrak{z})$. This implies that $m_\gamma(\beta_k)$ is a constant, where γ is a root of L defining Q . Since every root of $\Phi(\mathfrak{z})$ may be connected to the highest root of $\Phi(\mathfrak{g}/\mathfrak{p})$, this highest root lies in $\Phi(\mathfrak{z})$, \mathfrak{z} is an irreducible representation of L_Q , and $m_\gamma(\beta)$ is constant for $\beta \in \Phi(\mathfrak{z})$, where γ is a root of L defining Q , and this constant value is $m_\gamma(\mathfrak{g})$. \square

Corollary A.9. *We have the R -module isomorphism $Z(\mathfrak{n}_R)^* \simeq \mathfrak{z}$.*

Proof. The dual $Z(\mathfrak{n}_R)^*$ of the center of \mathfrak{n}_R is spanned by the vectors v_β for which $m_\gamma(\beta) = m_\gamma(\mathfrak{g})$ for all simple roots $\gamma \in \Phi(\mathfrak{n}_R)$. Thus by Lemma A.8 we have an injective R -module morphism from \mathfrak{z} to $Z(\mathfrak{n}_R)^*$. Since both \mathfrak{z} and $Z(\mathfrak{n}_R)$ are irreducible L_R -modules, this is an isomorphism. \square

Let G' be the subgroup $G' := Z_G(Z_H(Z(N_R)))$, $P' := G' \cap R$, and let L' be the Levi subgroup of P' . Note that $G' \supset H$, so that G' is determined by the weights $\Phi(\mathfrak{g}')$ of its Lie algebra.

Lemma A.10. $\Phi(\mathfrak{g}') = \mathbb{Q}\Phi(\mathfrak{z}) \cap \Phi(\mathfrak{g}) \subset \{\gamma \in \Phi(\mathfrak{g}) \mid \langle \gamma, \alpha_i \rangle = 0 \text{ for } i = 1, \dots, r\}$.

Proof. First note that $Z_{\mathfrak{h}}(Z(\mathfrak{n}_R)) = \Phi(Z(\mathfrak{n}_R))^\perp = \Phi(\mathfrak{z})^\perp$. Also, for any subalgebra $\mathfrak{h}' \subset \mathfrak{h}$, $\Phi(Z_{\mathfrak{g}}(\mathfrak{h}')) = (\mathfrak{h}')^\perp \cap \Phi(\mathfrak{g})$. Thus

$$\Phi(\mathfrak{g}') = (\Phi(\mathfrak{z})^\perp)^\perp \cap \Phi(\mathfrak{g}) = \mathbb{Q}\Phi(\mathfrak{z}) \cap \Phi(\mathfrak{g}),$$

proving the equality. The inclusion is a consequence of Lemma A.4. \square

Lemma A.11. $L' \subset Z_H(\mathfrak{z})\text{Stab}_L(\mathbb{C}v)$.

Proof. First note that $\Phi(\text{stab}_{\mathfrak{l}}(\mathbb{C}v)) = \{\gamma \in \Phi(\mathfrak{l}) \mid \langle \gamma, \alpha_i \rangle = 0 \text{ for } i = 1, \dots, r\}$, which contains $\Phi(\mathfrak{g}' \cap \mathfrak{l}) = \Phi(\mathfrak{l}')$, by Lemma A.10. Thus it suffices to show that $H \subset Z_H(\mathfrak{z})\text{Stab}_H(\mathbb{C}v)$. But since $\mathbb{Q}\phi(\mathfrak{z})$ and $\mathbb{Q}\{\alpha_1, \dots, \alpha_r\}$ are orthogonal, their annihilators, $\Phi(\mathfrak{z})^\perp = Z_{\mathfrak{h}}(\mathfrak{z})$ and $\{\alpha_1, \dots, \alpha_r\}^\perp \subset \text{stab}_{\mathfrak{h}}(\mathbb{C}v)$ together span \mathfrak{h} . \square

Lemma A.12. *The semisimple part of G' is simple.*

Proof. For any subset $\Gamma \subset \Phi(\mathfrak{g}')$, we form a graph by joining $\gamma_1, \gamma_2 \in \Gamma$ by an edge if $\langle \gamma_1, \gamma_2 \rangle \neq 0$. It suffices to show that there is a subset Γ of $\Phi(\mathfrak{g}')$ which spans $\mathbb{Q}\Phi(\mathfrak{g}')$ such that this graph is connected. We show that $\Gamma = \Phi(\mathfrak{z})$ is such a subset.

Extend $\alpha_1, \dots, \alpha_r$ to a maximal orthogonal sequence of long roots $\alpha_1, \dots, \alpha_m$. By Lemma A.5(ii) any pair $\alpha_i, \alpha_j \in \Phi(\mathfrak{z})$ have a common non-orthogonal root $\beta \in \Phi(\mathfrak{g}/\mathfrak{p})$. By Lemma A.5(i), β is orthogonal to $\alpha_1, \dots, \alpha_r$, hence in $\Phi(\mathfrak{z})$. Finally, every root $\beta \in \Phi(\mathfrak{z})$ is non-orthogonal to some α_i (necessarily in $\Phi(\mathfrak{z})$). Indeed, as $\alpha_1, \dots, \alpha_m$ is maximal, $\mathfrak{g}/\mathfrak{p} = \mathfrak{l} \cdot (v_{\alpha_1} + \dots + v_{\alpha_m})$. Then, by Lemma A.4,

$$\{\beta \in \Phi(\mathfrak{g}/\mathfrak{p}) \mid \langle \beta, \alpha_i \rangle \geq 1 \text{ for some } i\} = \Phi(\mathfrak{l} \cdot (v_{\alpha_1} + \dots + v_{\alpha_m})) = \Phi(\mathfrak{g}/\mathfrak{p}). \quad \square$$

Lemma A.13. *The nilradical $\mathfrak{n}_{P'}$ is equal to $Z(\mathfrak{n}_R)$, the center of the nilradical of R . In particular G'/P' is cominuscule.*

As the Killing form on \mathfrak{g}' identifies $(\mathfrak{n}_{P'})^*$ with $\mathfrak{g}'/\mathfrak{p}'$, this identifies \mathfrak{z} with $\mathfrak{g}'/\mathfrak{p}'$.

Proof. By our definition of P' , $\mathfrak{n}_{P'} \subset \mathfrak{n}_R$, and $\mathfrak{n}_{P'}$ is an H -module. By Lemma A.10, $Z(\mathfrak{n}_R) = \mathfrak{z}^* \subset \mathfrak{n}_{P'}$. Let $\gamma \in \Phi(\mathfrak{n}_{P'})$ be a weight that is not a weight of $Z(\mathfrak{n}_R)$. Then $-\gamma$ is either in $\Phi(\mathfrak{l}/\mathfrak{q})$ or else in $\Phi(\mathfrak{l} \cdot v)$, and thus there is some $i = 1, \dots, r$ such that $\langle -\gamma, \alpha_i \rangle \neq 0$, by Table 2. In particular, $-\gamma \notin \mathbb{Q}\Phi(\mathfrak{z})$, and so is not a weight of G' . As $N'_P = Z(N_R)$ is abelian, we deduce that G'/P' is cominuscule. \square

Lemma A.14. *For every $q \in R$, there exists $l \in L' \cap \text{Stab}_L(\mathbb{C}v)$ such that for every $z \in \mathfrak{z}$ we have $qz = lz$.*

Proof. First, we show that if $\gamma \in \Phi(\mathfrak{r}) - \Phi(\mathfrak{p}')$, then the weight vector v_γ acts trivially on \mathfrak{z} . This weight γ does not lie in $\mathbb{Q}\Phi(\mathfrak{z})$, and thus if $\beta \in \Phi(\mathfrak{z})$, then $\beta + \gamma \notin \Phi(\mathfrak{z})$. Since \mathfrak{z} is an \mathfrak{r} -module, this implies that $v_\gamma \cdot v_\beta = 0$.

It follows that there exists $p \in P'$ such that $pz = qz$ for every $z \in \mathfrak{z}$. However, as G'/P' is cominuscule, $N_{P'}$ acts trivially on its Lie algebra, and hence acts trivially on \mathfrak{z} , so we can replace p by an element of L' . Finally as $Z_H(\mathfrak{z})$ acts trivially on \mathfrak{z} , by Lemma A.11 we can reduce further to $L' \cap \text{Stab}_L(\mathbb{C}v)$. \square

Let $v' \in \mathfrak{g}'/\mathfrak{p}' = \mathfrak{z}$. We assume that v' is of the form $v' = v_{\alpha_{r+1}} + \cdots + v_{\alpha_{r+r'}}$, where $\alpha_{r+1}, \dots, \alpha_{r+r'}$ is an orthogonal sequence of long roots. Set $v_1 := v + v'$. Define $Q' := \text{Stab}_{L'}(\mathfrak{l} \cdot v')$, and $Q_1 = \text{Stab}_L(\mathfrak{l} \cdot v_1)$, and let R' and R_1 be the corresponding parabolic subgroups in P' and P , respectively.

Lemma A.15. $Q' = L' \cap Q_1$.

Proof. This follows from the characterization of $\Phi(\mathfrak{q}')$ and $\Phi(\mathfrak{q}_1)$ of Lemma A.6 as $\langle \gamma_1, \alpha_i \rangle = 0$, for $\gamma \in \Phi(\mathfrak{l}')$ and each $i = 1, \dots, r$. \square

Lemma A.16. $Z(\mathfrak{n}_{R_1}) = Z(\mathfrak{n}_{R'})$.

Proof. Note that the weights of $Z(\mathfrak{n}_{R_1})$ are exactly those in $\Phi(\mathfrak{n}_P)$ which are annihilated by α_i for $i = 1, \dots, r'$. The weights in $Z(\mathfrak{n}_{R'})$ are weights in $\Phi(\mathfrak{n}_{P'})$ which are annihilated α_i for $i = r + 1, \dots, r'$. Since $\Phi(\mathfrak{n}_{P'})$ are the weights of \mathfrak{n}_P annihilated by α_i for $i = 1, \dots, r$, the result follows. \square

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